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ABSTRACT

This sixteenth unit in the SMSG secondary school mathematics series is the teacher's commentary for Unit 14. For each of the chapters in Unit 14, a guide to the selection of problems is provided, the goals for that chapter are discussed, the mathematics is explained, some teaching suggestions are given, the answers to exercises are listed, and sample test questions for that chapter are included. A final section, labelled "Talks to Teachers," discusses facts and theories; equality, congruence, and equivalence; the concept of congruence; introduction to non-Euclidean geometry; miniature geometries; and area. (DT)

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TEACHER'S COMMENTARY UNIT NO. 16

GEOMETRY

PART II

SCHOOL MATHEMATICS STUDY GROUP

School Mathematics Study Group

Geometry

Unit 16

Geometry

Teacher's Commentary, Part II

REVISED EDITION

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A GUIDE TO THE SELECTION OF PROBLEMS

Following is a tabulation of the problems in this text. It will be noted that the problems are arranged into three sets, I, II, and III. At first glance, one might think that these are in order of difficulty.

THIS IS NOT THE MANNER IN WHICH THE PROBLEMS ARE GROUPED !!!!

Before explaining the grouping, it should be mentioned that it is understood that a teacher will select from all of the problems those which he or she feels are best for a particular class. However, careful attention should be given to the comments on the problems in *A Word About the Problem Sets*.

Group I contains problems that relate directly to the material presented in the text.

Group II contains two types of problems: (1) some that are similar to those of Group I, and (2) some that are just a little more difficult than those in Group I. A teacher may use this group for two purposes: (1) for additional drill material, if needed, and (2) for problems a bit more challenging than those in Group I, that could be used by a better class.

Group III contains problems that develop an idea, using the information given in the text as a starting point. Many of these problems are easy, interesting and challenging. The student may find them more stimulating than the problems in Groups I or II. However, if time is a factor, a student can very well not do any of them and still completely understand the material in the text. These are enrichment problems.

It is assumed that a teacher will not feel that he or she must assign all of the problems in any set, or all parts of any one problem. It is hoped that this listing will be helpful to you in assigning problems for your students.

We have included in the problem sets results of theorems of the text which are important principles in their own right. In this respect we follow the precedent of most geometry texts. However, all essential and fundamental theorems are in the text proper. The fact that many important and delightful theorems are to be found in the problem sets is very desirable as enrichment.

While no theorem stated in a problem set is used to prove any theorem in the text proper, they are used in solving numerical problems and other theorems in the problem sets. This seems to be a perfectly normal procedure. The difficulty (or danger), as most teachers define it, is in allowing the result of an intuitive type problem, or a problem whose hypothesis assumes too much, to be used as a convincing argument for a theorem. The easiest and surest way to handle the situation is to make a blanket rule forbidding the use of any problem result to prove another. Such a rule, however, tends to overlook the economy of time and, often, the chance to foster the creative spirit of the student. In this text we have tried to establish a flexible pattern which will allow a teacher and class to set their own policy.

GUIDE TO SELECTION OF PROBLEMS

	I	II	III
Chapter 11			
Set 11-1	1,2,3,6.	4,5.	7,8.
11-2	1,2,4,6,7,9, 10,11,16,18.	3,5,8,12,14,15, 19,20.	13,17,21,22.
11-3a	1,4,5,13,14.	2,3,8,11,12,15, 18,19.	6,7,9,10,16,17.
11-3b	1,2,4,6,7,14, 17,18,22,27.	3,5,8,9,10,11, 12,15,16,19,20, 21,23,24.	13,25,26,28.
Chapter 12			
12-1	1,2,3,4,5,6, 7,8,9,11.	10,12.	
12-2	1,2,4,5.		3.
12-3a	1,2,3,4,5,6, 11,12.	7,8,9,10,13.	14.
12-3b	1,2,3,4,5,6,7, 13,14,23,24.	8,9,10,12,18,19, 21,22,25,26,30, 31.	15,16,17,20,27, 28,29,32.
12-4	1,2,3,4,5.		
12-5	1,2,3,4,5,6, 7,8,13,15,17.	9,10,11,12,14, 18.	16,19,20.
Chapter 13			
13-2	1,2,3,4,5,6, 8,9,13,15	7,10,11,16,17, 19.	12,14,18.
13-3	1,2,3,4.	5,8,9,10.	6,7.

	I	II	III
Chapter 13			
Set 13-4a	1,2,3,4,5,6, 7,9,10,11.	8.	12,13.
13-4b	1,2,4,8,9,10, 11,16.	3,5,6,7,12, 13,14.	15.
13-5	1,2,3,4,7,13, 16.	5,6,8,9,10, 12,14,15,17, 18.	11,19.
Chapter 14			
14-1	1,2,3,4,5,6, 7.		
14-2a	1,2,3,4,6,7, 8.	5,9.	10,11.
14-2b	1,2,5,6,8.	3,4,7.	
14-3	1,4.	2,3.	
14-5a	1,2,3,4,5,6, 7.		
14-5b	1,2,3,4.	5,6.	
14-5c	1,3,5.	6,7,8.	2,4,9,10,11,12.
14-7	1,2.		3,4,5,6.
Chapter 15			
15-1	1,2,5.	6.	3,4.
15-2	1,2.4,5,11.	3,7,12.	6,8,9,10.
15-3	1,3,4,7.	5,6.	2,8.
15-4	1,2,3,4,5.	6,7,8,9,11,14.	10,12,13.
15-5	1,2,3,5.	4,6,7,8.	

	I	II	III
Chapter 16			
Set 16-1	1,2,3,4,5,6.		
16-2	1,2,3,5,6,7.	8.	4.
16-3	1,2,3,4,5,6.	7.	8.
16-4	1,2,3,4,5,6.	7,8.	9.
16-5	1,2,3,4,7,8.	5,6,9,10.	11.
Chapter 17			
17-3	2,3,4,5,6,7,8,9.		10,11,12.
17-4	1,2,3,4,5,7,8,9,12.	6,10,13,14,15.	11,16.
17-5	1,2,5,6,8.	3,4,7.	9,10,11.
17-6	1,2,3,4,5,6,7,9.	8.	10.
17-7	1,2,3,4,5,7.	6.	8,9.
17-8	1,2,3,4,5.	6,7,9,10,11,	8.
17-9	1,2,3,4,5,6.	7,10.	8,9,11,12,13,14,15.
17-10	1,2,3,5,6,7,8.	4,9,10,11,12,13,14.	15,16,17,18.
17-12	1,2,3,4,13,16,17,18,19.	9,10,14,15.	5,6,7,8,11,12,20.
17-13	1,2,6.	3,7.	4,5,8,9.
17-14	1,2.	3,4,5.	6,7,8,9.

Chapter 11

AREAS OF POLYGONAL REGIONS

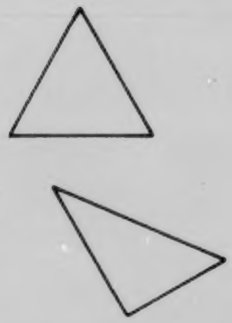
This Chapter treats the conventional subject matter of the areas of triangles, parallelograms, trapezoids and so on. Although its viewpoint is essentially that of Euclid two points may seem novel. First the introduction of the term polygonal region and second the study of area by postulating its properties rather than by deriving them from a definition of area based on the measurement process. Actually both of these ideas are implicit in the conventional treatment - we have only brought them to the surface and sharpened and clarified them. Once the basis has been laid, our methods of proof are simple and conventional, although the order of the theorems may seem a bit unusual.

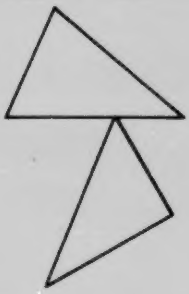
317 Observe that in this Chapter we are not trying to develop a very general theory of area applicable for example to figures with curvilinear boundaries. Rather we restrict ourselves to the relatively simple case of a region whose boundary is rectilinear, that is, its boundary is a union of segments. However, it is not obvious how to define the concept of region or of boundary. One suggestion is to turn the problem around and merely consider the figure composed of a polygon and its interior. However, although there is no essential difficulty in defining polygon (see Section 15-1 of text) it is quite difficult to write down precisely a definition of the interior of a polygon, even though we can easily test in a diagram whether or not a point is in the interior of a polygon. Observe how simply our definition of polygonal region avoids this difficulty. We merely take the simplest and most basic type of region, the triangular region, and use it as a sort of building block to define the idea of polygonal region. The essential point is, that, although it is difficult to define interior for an arbitrary polygon, it is very easy to do it for a triangle - we actually did this back in Chapter 4. Moreover

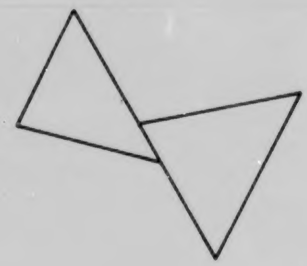
our basic procedure in studying area is to split a figure into triangular regions, and reason that its area is the sum of the areas of these triangular regions. Thus we simply define polygonal regions as figures that can be suitably "built up" from triangular regions, and we have a good basis for our theory.

318 A further point. The definition requires that the triangular regions must not "overlap", that is they must not have a triangular region in common (see the discussion in the text following the definition of polygonal region), but may have only a common point or a common segment. If we permit the regions to "overlap" we can't say that the area of the whole figure will be the sum of the areas of its component triangular regions (see discussion in the text following Postulate 19). Thus for simplicity we impose the condition that the triangular regions shall not "overlap".

319 A final point. In your intuitive picture of a polygonal region you probably have assumed that a polygonal region is connected or "appears in one piece". Actually our definition does not require this. It permits a polygonal region to be the union of two triangular regions which have no point (or one point or a segment) in common, as in these figures:







Thus our definition allows a polygonal region to be a disconnected portion of the plane, and the boundary of a polygonal region need not be a single polygon. This causes no trouble - it just means that our theory has somewhat broader coverage than our intuition suggests.

In light of this you will note that the idea of polygon is not emphasized as strongly in our text as in the conventional treatment. When the latter refers to "area of a polygon" it means the area of the polygonal region consisting of the polygon and its interior - which is not explicitly stated or clarified. We avoid the difficulty by defining polygonal region independently of polygon.

319 Note that in the figures on page 256 it is intuitively clear that the areas of the regions can be found by dividing them up into smaller triangular regions, and that the area of the total region is independent of the manner in which the triangular regions are formed.

Sometimes in a mathematical discussion we give an explicit definition of area for a certain type of figure. For example, the area of a rectangle is the number of unit squares into which the corresponding rectangular region can be separated. This is a difficult thing to do in general terms for a wide variety of figures. Thus the suggested definition of area of a rectangle (rectangular region) is applicable only if the rectangle has sides whose lengths are integers. Literally how many unit squares are contained in a rectangular region whose dimensions are $\frac{1}{2}$ and $\frac{1}{3}$? The answer is none! Clearly the suggested definition must be modified for a rectangle with rational dimensions. To formulate a suitable definition when the dimensions are irrational numbers, say $\sqrt{2}$ and $\sqrt{3}$, is still more complicated and involves the concept of limits. Incidentally, even when this is done, it would not be trivial to prove that the area of such a rectangle is given by the familiar formula. (For example, see the Talk on Area.) Furthermore, it would still be necessary to define the area concept for triangles, quadrilaterals, circles, and so on. The complete study of area along these lines involves integral calculus and finds its culmination in the branch of modern mathematics called the Theory of Measure. (See the Talk on Area for a treatment of area in the spirit of the theory of measure.)

Clearly this is too heroic an approach for our purposes. So we don't attempt to give an explicit definition of area of a polygonal region by means of a measurement process using unit squares. Rather we study area in terms of its basic properties as stated in Postulates 17, 18, 19 and 20. On the basis of these postulates we prove the familiar formula for the area of a triangle (Theorem 11-2). Consequently we get an explicit procedure for obtaining areas of triangles and so of polygonal regions in general.

319 Some remarks on the postulates. Observe that our treatment of area is similar to that for distance and measure of angles. Instead of giving an explicit definition of area (or distance or angle measure) by means of a measurement process, we postulate its basic properties which are intuitively familiar from study of the measurement process.

320 Thus Postulate 17 asserts that to every polygonal region there is associated a unique "area number" and is exactly comparable to the Distance Postulate or the Angle Measurement Postulate. The uniqueness of the area number is based on the intuitive presupposition that a fixed unit has been chosen and that we know how to measure area in terms of this unit.

320 Postulate 18 is one of the simplest and most natural properties of area. If two triangles are congruent then in effect the triangular regions determined are "congruent", one is an exact replica of the other, and so they must have the same measure.

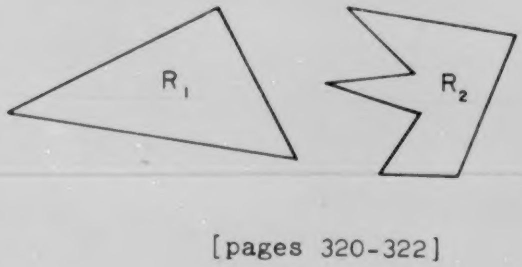
320 Postulate 19 is comparable to the Angle Addition Postulate. It is a precise formulation, for the study of area, of the vague statement "The whole is the sum of its parts". This statement is open to several objections. It seems to mean that the measure of a figure is the sum of the measures of its parts. Even in this form it is not acceptable, since the terms "figure" and "part" need to be sharpened in this context, and it permits the "parts" to overlap. Postulate 19 makes clear that the "figures" are to be polygonal regions, the "measures" are areas, and that the "parts" are to be polygonal regions whose union is the "whole" and which do not overlap.

Postulates 17, 18 and 19 seem to give the essential properties of area, but they are not quite complete. We pointed out above that Postulate 17 presupposes that a unit has been chosen, but we have no way of determining such a unit, that is, a polygonal region whose area is unity. For example, Postulates 17, 18 and 19 permit a rectangle of dimensions 3 and 7 to have area unity.

322 Postulate 20 takes care of this by guaranteeing that a square whose edge has length 1 shall have area 1. In addition, Postulate 20 gives us an important basis for further reasoning by assuming the formula for area of a rectangle.

An interesting point: We could have replaced Postulate 20 by the assumption of the familiar formula for the area of a triangle. This is equivalent to Postulate 20.

The use of the term "at most" in Postulate 19 permits R_1 and R_2 to have no common point, as in this figure:



Since we are introducing a block of postulates concerning area, this may be a good time to remind your students of the significance and purpose of postulates. They are precise formulations of the basic intuitive judgments suggested by experience, from which we derive more complex principles by deductive reasoning.

To make Postulates 17, 18 and 19 significant for the students, discuss the measuring process for area concretely, using simple figures like rectangles or right triangles with integral or rational dimensions. Have them subdivide regions into congruent unit squares, so that the student gets the idea that every "figure" has a uniquely determined area number. Then present the postulates as simple properties of the area number which are verifiable concretely in diagrams.

Problem Set 11-1

323 1. a. 2, d. 4,
b. 2, e. 6.
c. 5,

324 2. 825 square feet.

3. a. The area is doubled.

b. The area is four times as great.

4. 1800 tiles.

5. 792 square inches.

325 *6. a. False. A triangle is not a region at all, but is a figure consisting of segments.

b. False. See Postulate 17.

c. True. By Postulate 17.

d. True. By Postulate 18.

e. False. If the regions overlap, their union is less than their sum.

f. True. Since a square is a rectangle.

g. False. The region is the union of a trapezoid and its interior.

h. True. A triangular region is the union of one or more triangular regions.

326 7. a. 4.

b. $\frac{1}{4}$.

c. 0.1.

d. 0.002.

e. and f. Since $\sqrt{2}$ and $\sqrt{3}$ are irrational, the base and altitude in each case do not have a common divisor. Hence the rectangular regions cannot be divided exactly into squares.

*8. a. $f - e + v = 7 - 12 + 7 = 2$.

b. $f - e + v = 7 - 17 + 12 = 2$.

c. The computation always results in 2.

d. The computation is not affected, since the additional four edges, three faces, and one vertex results in zero being added to the total.

e. No change.

---

328 Notice that, after postulating the area of a rectangle, we proceed to develop our formulas for areas in the following manner: right triangles, which then permit us to work with any triangle, parallelograms, and trapezoids. Of course our postulate permits us to find the area of a square, since it is merely an equilateral rectangle. At this point we have the machinery to find the area of any polygonal region, by just chopping it up into a number of triangular regions, and

finding the sum of the areas of these triangular regions.

Note that in the discussion of the area of a triangle, it does not matter which altitude and base we consider, just so long as we work with a base and the corresponding altitude.

In the application of Postulate 19 to a specific case we read from a figure that R is the union of the regions R_1 and R_2; see for example the proofs of Theorems 11-1 and 11-2. This is a kind of separation theorem which can be justified from our postulates. Just as with triangles, we may work with either side and the corresponding altitude of a parallelogram.

In Problem Set 11-2, Problems 13-17 form a sequence of problems involving an interesting consequence of the theorems of the text.

Problem Set 11-2

333 1. a. Area Δ ABC $= \frac{1}{2}\cdot 7\cdot 24 = 84$.

b. $84 = \frac{1}{2}\cdot 25h$. $h = 6\frac{18}{25}$.

2. 14.4 and 24.

3. a. BC = 12. c. AB = 15.

b. CD $= 6\frac{9}{11}$. d. AE $= \frac{ch}{a}$.

4. Area Δ CQB = Area Δ DQB, since CQ = DQ and the triangles have the same altitude, the perpendicular segment from B to $\overline{CD}$. Area Δ AQC = Area Δ DQA, since CQ = DQ and the triangles have the same altitude, the perpendicular segment from A to $\overline{CD}$. Adding, we have Area Δ ABC = Area Δ ABD.

Alternate Proof: Draw $\overline{CE} \perp \overline{AB}$ and $\overline{DF} \perp \overline{AB}$. Then Δ CEQ $\cong \Delta$ DFQ by A.A.S., and CE = DF. Since Δ ABC and Δ ABD have the same base and their altitudes have equal lengths, the triangles have equal areas.

33^4 5. The area of the square is s^2.
The area of each of the four triangles is $\frac{1}{2}bs$.
Hence, the area of the star is $s^2 - 2bs$.

6. a. 6.

b. 12.

c. $18\frac{2}{3}$.

d. Since GB and AF are measures of the same altitude, there is not enough information given to determine a unique answer.

7. Since a diagonal of a parallelogram divides it into two congruent triangles, Area Δ AFH is equal to half the area of the parallelogram. Area Δ AQH = Area Δ FQH since the bases, $\overline{AQ}$ and $\overline{QF}$, are congruent and the triangles have the same altitude, a perpendicular from H to $\overline{AF}$. Each is then one-fourth of the area of the parallelogram. In the same way it can be shown that Area Δ ABQ = Area Δ FBQ.

335 8. a. 36.

b. 21.

c. 55.

d. $136\frac{1}{2}$.

e. $121\frac{1}{2}$.

9. 98.

10. Area of triangle = $\frac{1}{2}bh$.

Area of parallelogram = bh'.

$\frac{1}{2}bh = bh'$.

$h = 2h'$.

The altitude of the triangle is twice the altitude of the parallelogram.

11. a. Area parallelogram ABCD is twice area Δ BCE because the figures have the same base ($\overline{BC}$) and equal altitudes, since $\overline{AE} \parallel \overline{BC}$.

335 11. b. The two areas are equal.

c. The areas are equal because the bases ($\overline{AF}$ and $\overline{FD}$) are congruent and their altitudes are congruent since $\overline{AD} \parallel \overline{BC}$.

d. Area Δ CFD = $\frac{1}{2}$(area Δ BCE) since FD = $\frac{1}{2}$BC and the two triangles have equal altitudes. Therefore, area parallelogram ABCD = 2(area Δ BCE) = 4(area Δ CFD).

336 12. The area of trapezoid DFEC = 34.
The area of trapezoid AGFD = 165.
And so, area of AGECD = 199.
Area Δ AGB = 30.
Area Δ BCE = $32\frac{1}{2}$.

Subtracting the sum of the areas of the two triangles from the area of AGECD, we have $136\frac{1}{2}$. The area of the field is $136\frac{1}{2}$ square rods.

13. Given: Figure ABCD with $\overline{AC} \perp \overline{DB}$.

Prove: Area of ABCD = $\frac{1}{2}$AC·DB.

Proof: Area of ABCD = Area Δ ACD + Area Δ ABC by Postulate 19.

But Area Δ ACD = $\frac{1}{2}$AC·DP and Area Δ ABC = $\frac{1}{2}$AC·PB.

Therefore, Area of ABCD = $\frac{1}{2}$AC·DP + $\frac{1}{2}$AC·PB

$= \frac{1}{2}AC(DP + PB) = \frac{1}{2}AC \cdot DB.$

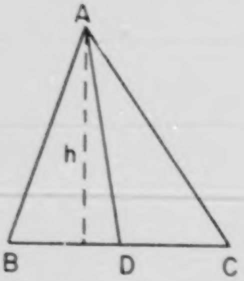
14. The area of a rhombus equals one-half the product of the lengths of its diagonals.

15. 12.

16. A = $\frac{1}{2}$dd' = 150 = bh = 12b; therefore b = $12\frac{1}{2}$. The area is 150; the length of a side is $12\frac{1}{2}$.

*17. Yes. The proof would be the same as for Problem 13 with each "+" replaced by "-".

337 18. All three triangles have the same altitude. Hence, since BD = DC, the two smaller triangles have equal area, by Theorem 11-6, and each is one-half the area of the big triangle, by Theorem 11-5.



19. a. By the previous problem,

Area Δ ABE = Area Δ BAD = $\frac{1}{2}$(Area Δ ABC). Subtracting Area Δ ABG from each, leaves Area Δ AEG = Area Δ BDG.

b. Since the medians are concurrent, the third median, with the other two, divides the triangle into six triangles:
Area Δ AEG = Area Δ BDG,
Area Δ CGE = Area Δ BGF, and
Area Δ CGD = Area Δ AGF. But Area Δ BDG = Area Δ CGD by Theorem 11-6, and consequently all the areas are equal. Therefore,
Area Δ BDG = $\frac{1}{6}$(Area Δ ABC).

20. Since AB is constant, the altitude to $\overleftrightarrow{AB}$ must be constant, by Theorem 11-6.
Call the length of the altitude, from P to $\overleftrightarrow{AB}$, h. Then in plane E, P may be any point on either of the two lines parallel to $\overleftrightarrow{AB}$ at a distance h from $\overleftrightarrow{AB}$. In space, P may be any point on a cylindrical surface having $\overleftrightarrow{AB}$ as its axis and h as its radius.

338 *21. a. 104.

b. $\frac{1}{2} \cdot 16 \cdot 13 = 104$.

c. With the dimensions given ABN and ADE would not be straight segments, and so the figure would not be a triangle.

338 *22. If the line intersects adjacent sides, the area of the triangle formed will be less than one-half the area of the rectangle, so the line must intersect opposite sides.

$$\text{Area } ARSD = \frac{1}{2}h(a + c)$$

$$\text{Area } CSRB = \frac{1}{2}h(b + d).$$

$$a + c = b + d.$$

But $a + b = c + d$, so by subtraction,

$$c - b = b - c,$$

$$c = b.$$

Let M be the point at which $\overline{AC}$ intersects $\overline{RS}$. Then $\Delta\ ARM \cong \Delta\ CSM$ by A.S.A., so $AM = CM$. Therefore M is the mid-point of diagonal $\overline{AC}$.

---

339 We have here a very simple proof of the Pythagorean Theorem. The proof depends upon the properties of the areas of triangles and squares. Notice how Postulate 19 is used in this proof.

Observe that the proof is perfectly general. The Pythagorean relation is proved for the sides of the constructed triangle and so holds for the original triangle.

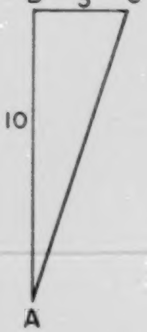
Problem Set 11-3a

341 1. $(AC)^2 = 100 + 9.$

$= 109.$

$AC = \sqrt{109}.$

He is $\sqrt{109}$ miles from his starting point. (Between 10.4 and 10.5 miles.)

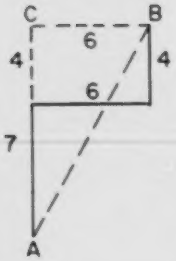


341 2. The single right triangle ΔACB serves our purpose here.

$(AB)^2 = (11)^2 + (6)^2 = 157.$

$AB = \sqrt{157}.$

He is approximately 12.5 miles from his starting point.

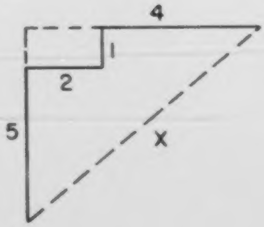


3. $(6)^2 + (6)^2 = x^2.$

$72 = x^2.$

$6\sqrt{2} = x.$

He is approximately 8.5 miles from his starting point.



4. In right $\Delta\ ABC$, $(AC)^2 = (4)^2 + (12)^2 = 16 + 144 = 160$. $AC = \sqrt{160} = 4\sqrt{10}$. In right $\Delta\ ACD$, $(AD)^2 = 160 + 9 = 169$. $AD = 13$.

Or, in $\Delta\ ABE$, $(AE)^2 = (4)^2 + (3)^2 = 16 + 9 = 25$. $AE = 5$. in $\Delta\ AED$, $(AD)^2 = (5)^2 + (12)^2 = 25 + 144 = 169$. $AD = 13$.

5. a, c, d, e.

342 6. a. It is sufficient to show that $(m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2$. $(m^2 - n^2)^2 + (2mn)^2 = m^4 - 2m^2n^2 + n^4 + 4m^2n^2 = m^4 + 2m^2n^2 + n^4 = (m^2 + n^2)^2$.

b. $m = 2$, $n = 1$ gives sides with lengths (3, 4, 5).
$m = 3$, $n = 1$ gives (6, 8, 10).
$m = 3$, $n = 2$ gives (5, 12, 13).
$m = 4$, $n = 1$ gives (15, 8, 17).
$m = 4$, $n = 2$ gives (12, 16, 20).
$m = 4$, $n = 3$ gives (7, 24, 25).

There are two other right triangles with hypotenuse less than or equal 25, (9, 12, 15) and (15, 20, 25), but they can not be obtained by this method.

342 7. a. $AY = \sqrt{2}$, $AZ = \sqrt{3}$. $AB = \sqrt{4} = 2$.

b. $AC = \sqrt{5}$. Next segment has length $= \sqrt{6}$.

8. $AC = \sqrt{8}$ or $2\sqrt{2}$.

$(AY)^2 = (AC)^2 + (YC)^2$, from which $AY = 3$.

343 *9. a. $h_c{}^2 = 13^2 - x^2 = 169 - x^2$;

also $h_c{}^2 = 15^2 - (14 - x)^2 = 225 - 196 + 28x - x^2$.

Eliminating $h_c{}^2$:

$169 - x^2 = 29 + 28x - x^2$.

$28x = 140$.

$x = 5$,

$h_c = 12$.

b. $h_a = 14^2 - x^2 = 196 - x^2$;

also $h_a{}^2 = 13^2 - (15 - x)^2 = 169 - 225 + 30x - x^2$.

Or $AB \cdot h_c = BC \cdot h_a$

$14 \cdot 12 = 15h_a$

$11\frac{1}{5} = h_a$.

*10. Let $\overleftrightarrow{CD}$ meet $\overleftrightarrow{AB}$ at D. Let $BD = x$.

$h_c{}^2 = 14^2 - x^2 = 196 - x^2$,

also $h_c{}^2 = 18^2 - (6 + x)^2 = 324 - 36 - 12x - x^2$.

Eliminating $h_c{}^2$:

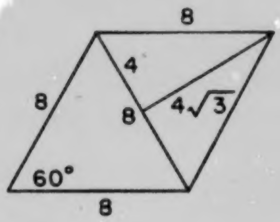
$196 - x^2 = 288 - 12x - x^2$.

$12x = 92$.

$x = 7\frac{2}{3}$.

$h_c = \frac{1}{3}\sqrt{1235}$. (approximately 11.71.)

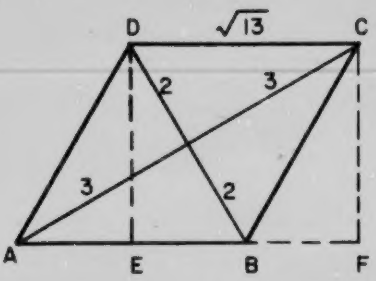
343 11. The shorter diagonal divides the rhombus into two equilateral triangles. Hence its length is 8. Since the diagonals are perpendicular bisectors of each other we can use the Pythagorean Theorem to get the length of the longer diagonal equal to $8\sqrt{3}$.



12. Since the sides are all congruent, and the area of the rhombus is the product of the measures of any side and its corresponding altitude, then all the altitudes are congruent. Hence, it is sufficient to find one altitude. The diagonals bisect each other at right angles. Hence, each side has length $\sqrt{13}$. Then,

Area of Δ ABD $= \frac{1}{2}\cdot 4\cdot 3 = 6 = \frac{1}{2}DE\sqrt{13}$,

and $DE = \frac{12}{13}\sqrt{13}$.



13. By the Pythagorean Theorem, AB = 13.

The area of Δ ABC $= \frac{1}{2}\cdot 13h = \frac{1}{2}\cdot 5\cdot 12$.

Hence $13h = 5\cdot 12$ and $h = \frac{60}{13} = 4\frac{8}{13}$.

344 14. By the Pythagorean Theorem, AB = 17.

The area of Δ ABC $= \frac{1}{2}\cdot 17h = \frac{1}{2}\cdot 15\cdot 8$.

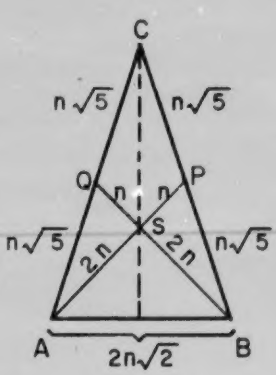
Hence $17h = 15\cdot 8$ and $h = \frac{120}{17} = 7\frac{1}{17}$.

344 15. Area Δ ABC = $\frac{1}{2}ch$, and

$$h = \frac{2(\text{Area } \Delta \text{ ABC})}{c}.$$ But Area Δ ABC = $\frac{1}{2}ab$,

and $c = \sqrt{a^2 + b^2}$. Therefore, $h = \frac{ab}{\sqrt{a^2 + b^2}}$.

*16. Lengths are shown in the figure.



Area Δ ASQ = $\frac{1}{2}(n \cdot 2n) = n^2$.

Area Δ ABS = $\frac{1}{2}(2n \cdot 2n) = 2n^2$.

Area Δ ABC = $\frac{1}{2}(3n\sqrt{2} \cdot 2n\sqrt{2})$

$= 6n^2$.

Area QSPC = Area Δ ABC

− (Area Δ ABS + 2 Δ ASQ)

$= 6n^2 - 4n^2 = 2n^2$.

17. Since Δ ABC $\cong$ Δ BED, m$\angle$ BAC = m$\angle$ EBD. But $\angle$ BAC is complementary to $\angle$ ABC, so $\angle$ EBD is complementary to $\angle$ ABC. Since $\angle$ EBD + $\angle$ EBA + $\angle$ ABC = 180, then $\angle$ EBA = 90. Now,

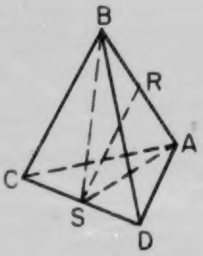
Area of CAED = Area Δ ABC + Area Δ AEB + Area Δ BED.

$\frac{1}{2}(a + b)(a + b) = \frac{1}{2}ab + \frac{1}{2}c^2 + \frac{1}{2}ab$.

$a^2 + 2ab + b^2 = 2ab + c^2$.

$a^2 + b^2 = c^2$.

345 *18. a. $\overline{SB}$ is a median of isosceles Δ BCD and therefore $\overline{SB} \perp \overline{CD}$. In the same way, $\overline{SA} \perp \overline{CD}$. $\therefore$ $\overline{CD} \perp$ plane BSA, and $\overline{CD} \perp \overline{SR}$. SB = SA (they are corresponding medians of congruent equilateral triangles). $\overline{SR}$ is a median to the base of isosceles Δ SBA and hence $\overline{SR} \perp \overline{BA}$.



345 b.

1. $DA = 2$.	1. Given.
2. $SD = 1$.	2. Definition of mid-point.
3. $SA = \sqrt{3}$.	3. Pythagorean Theorem.
4. $RA = 1$.	4. Definition of mid-point.
5. $SR = \sqrt{2}$.	5. Pythagorean Theorem.

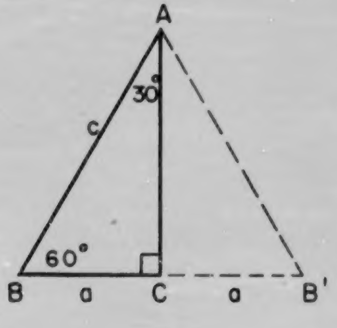
*19. By Pythagorean Theorem, $AC = \sqrt{2}$. Therefore $CD = \sqrt{2}$ and $BD = 1 + \sqrt{2}$. Hence, $(AD)^2 = 1 + (1 + \sqrt{2})^2 = 4 + 2\sqrt{2}$. Then $AD = \sqrt{4 + 2\sqrt{2}}$.

Since $AC = CD$, $m\angle ADC = m\angle CAD$. But $m\angle ADC + m\angle CAD = x = 45$. Then $2(m\angle ADC) = 45$, and $m\angle ADC = 22\frac{1}{2}$. $m\angle DAB = 67\frac{1}{2}$.

346

Proofs of Theorems 11-9 and 11-10

Theorem 11-9. (The 30-60 Triangle Theorem.) The hypotenuse of a right triangle is twice as long as a leg if and only if the measures of the acute angles are 30 and 60.



Restatement: Given ΔABC with $m\angle C = 90$, $AB = c$ and $BC = a$.

(1) If $m\angle A = 30$ and $m\angle B = 60$, then $c = 2a$.

(2) If $c = 2a$, then $m\angle A = 30$ and $m\angle B = 60$.

Proof: We begin in the same way for both parts. On the ray opposite to $\overrightarrow{CB}$ take B' such that $B'C = BC = a$. $\triangle BCA \cong \triangle B'CA$ by S.A.S. Then

(1) $m\angle B' = 60$ and $m\angle BAB' = 60$. Hence BAB' is equilateral so that $BB' = c = 2a$, which was to be proved.

(2) $AB' = AB = c$. By hypothesis, $c = 2a$. Since $BB' = 2a$, then $BB' = c$, and $\triangle BAB'$ is equilateral. Therefore $\triangle BAB'$ is equiangular and $m\angle B = 60$. Since $m\angle BCA = 90$, then $m\angle BAC = 30$, which was to be proved.

Note that we can now conclude that $\overline{BC}$, opposite the 30° angle is the shorter leg, since $m\angle A < m\angle B$. But before we had proved this inequality there was still the possibility that $\overline{AC}$ was the longer leg.

Since we know that $AC > BC$ it seems natural to derive their exact relationship. By the Pythagorean Theorem we have

$$(AC)^2 = c^2 - a^2,$$
$$(AC)^2 = (2a)^2 - a^2,$$
$$(AC)^2 = 3a^2.$$

Therefore, $AC = a\sqrt{3}$ or $AC = BC\sqrt{3}.$

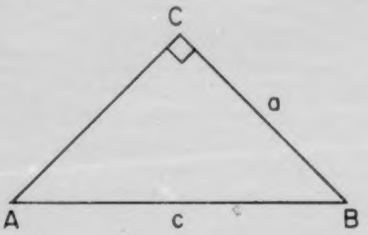
Using the above relationships for a 30-60 triangle we can always find all sides if we know one of the sides.

346 Theorem 11-10. (The Isosceles Right Triangle Theorem.) A right triangle is isosceles if and only if the hypotenuse is $\sqrt{2}$ times as long as a leg.

Restatement: Given $\triangle ABC$ with $m\angle C = 90$, $AB = c$ and $BC = a$.

(1) If $c = a\sqrt{2}$, then $\triangle ABC$ is isosceles.

(2) If $\triangle ABC$ is isosceles, then $c = a\sqrt{2}.$



Proof: (1) Using the Pythagorean Theorem,

$$(AC)^2 = c^2 - a^2,$$

$$(AC)^2 = (a\sqrt{2})^2 - a^2,$$

$$(AC)^2 = a^2,$$

$AC = a$, which was to be proved.

(2) Using the Pythagorean Theorem,

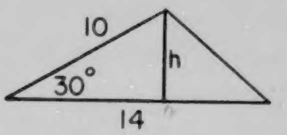
$$(AB)^2 = a^2 + a^2 = 2a^2,$$

$AB = a\sqrt{2}$, which was to be proved.

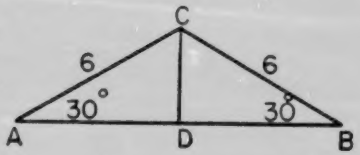
These two theorems suggest many useful facts in solving numerical problems. For example, in an equilateral triangle with side s the altitude is $\frac{s}{2}\sqrt{3}$ and its area is $\frac{s^2}{4}\sqrt{3}$. Certain of the problems in Problem Set 11-3b develop such ideas. Such key Problems are numbers 4, 7, and 17.

Problem Set 11-3b

346 1. 5. 35.



2. Draw $\overline{CD} \perp \overline{AB}$.
Then $AD = DB = 3\sqrt{3}$.
$AB = 6\sqrt{3}$.



347 3. Let x = the length of the shorter leg. Since the triangle is a $30^\circ - 60^\circ$ triangle,

$$(2x)^2 - x^2 = 75.$$

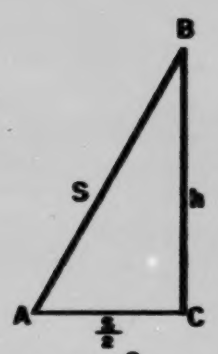
$$3x^2 = 75.$$

$$x^2 = 25.$$

$$x = 5.$$

[pages 346-347]

The length of the hypotenuse is 10.



347 4. By Theorem 11-9, $AC = \frac{s}{2}$

Since $(AC)^2 + (BC)^2 = (AB)^2$,

we have, $(\frac{s}{2})^2 + h^2 = s^2$

from which $h^2 = s^2 - \frac{s^2}{4}$.

$h^2 = \frac{3s^2}{4}$ so $h = \frac{s}{2}\sqrt{3}$.

5. Since $m\angle B = 60$, then $m\angle D = 60$, and $DF = \frac{3}{2}$.

Then $AF = \frac{3}{2}\sqrt{3}$.

6. $\frac{s}{2}\sqrt{3} = 15$.

$s = \frac{30}{\sqrt{3}}$.

$s = 10\sqrt{3}$. A side is $10\sqrt{3}$ inches long.

7. $\frac{1}{2}$; 2; $\frac{\sqrt{3}}{3}$; $\sqrt{3}$; $\frac{\sqrt{3}}{2}$ $\frac{2\sqrt{3}}{3}$; Yes.

8. a. $\frac{1}{2}$base $= 10\sqrt{3}$, altitude $= 10$.

$10 \cdot 10\sqrt{3} = 100\sqrt{3}$.

Area is $100\sqrt{3}$ square inches.

b. $\frac{1}{2}$base $= 10\sqrt{2}$, altitude $= 10\sqrt{2}$.

$10\sqrt{2} \cdot 10\sqrt{2} = 200$.

Area is 200 square inches.

c. $\frac{1}{2}$base $= 10$, altitude $= 10\sqrt{3}$.

$10 \cdot 10\sqrt{3} = 100\sqrt{3}$.

Area is $100\sqrt{3}$ square inches.

9. a. $\frac{1}{2}$base $= 12$, $h = 12$. Area is 144 square inches.

b. $\frac{1}{2}$base $= 12$, $h = 4\sqrt{3}$. Area is $48\sqrt{3}$ square inches.

c. $\frac{1}{2}$base $= 12$, $h = 12\sqrt{3}$. Area is $144\sqrt{3}$ square inches.

348 10. a. $a = 30.$
$2a = 60.$
$3a = 90.$
$x = 6.$
$y = 3\sqrt{3}.$

b. $a = 30.$
$x = 5\sqrt{3}.$
$y = 10.$

c. $a = 45.$
$2a = 90.$
$x = 5.$
$y = 5\sqrt{2}.$

d. $a = 45.$
$x = 4.$
$y = 4\sqrt{2}.$

349 e. $x = 2\sqrt{3}.$
$y = 4.$

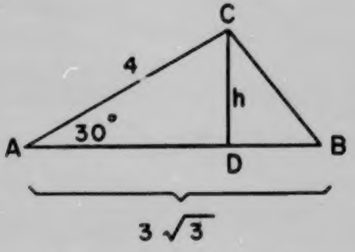
f. $x = 4\sqrt{2}.$
$y = 4\sqrt{2}.$

g. $a = 60.$
$x = 3\sqrt{3}.$

h. $a = 45.$
$x = 2\sqrt{2}.$

11. $FB = 3$; $HF = 3\sqrt{3}$; $AH = 6\sqrt{2}$; $AF = 3\sqrt{5}$; $m\angle ABF = 90$; $m\angle ABH = 90$; $m\angle HFB = 90$; $m\angle HBF = 60$; $m\angle BHA = m\angle BAH = 45$.

*12. Let $\overline{CD}$ be the altitude to $\overline{AB}$. Let $AD = x$, $CH = h$, $BC = a$, $DB = y$. In 30° - 60° right Δ ACD,
$h = \frac{1}{2}\cdot 4 = 2$, $x = 2\sqrt{3}$.
Therefore $y = 3\sqrt{3} - 2\sqrt{3}$
$= \sqrt{3}$. In right Δ DBC,
$a^2 = h^2 + y^2 = 4 + (\sqrt{3})^2 = 7.$
$a = \sqrt{7}.$
No, since $(4)^2 + (\sqrt{7})^2 \neq (3\sqrt{3})^2$.

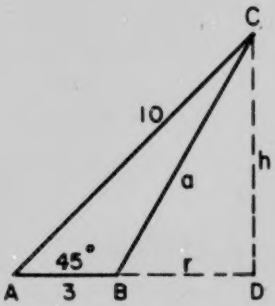


350 *13. Let $\overline{CD}$ be the perpendicular from C to $\overleftrightarrow{AB}$. Let CD = h, BD = r, BC = a.

In $45^\circ - 45^\circ - 90^\circ$ $\triangle$ ACD,

$h = AD = \frac{1}{2}\sqrt{2}\cdot 10 = 5\sqrt{2},$

$r = AD - 3 = 5\sqrt{2} - 3.$



In right $\triangle$ BDC,

$$a^2 = r^2 + h^2 = (5\sqrt{2} - 3)^2 + (5\sqrt{2})^2.$$

$$= 50 - 30\sqrt{2} + 9 + 50$$

$$= 109 - 30\sqrt{2}.$$

$a = \sqrt{109 - 30\sqrt{2}}.$ BC is approximately 8.2.

14. By Pythagorean Theorem, the altitude equals 24. The area is 240 square inches.

15.

1. $\triangle$s DFB and CFA are right triangles.	1. Given.
2. FD = FC. DB = CA.	2. Given.
3. $\triangle$ FDB $\cong$ $\triangle$ FCA.	3. Hypotenuse-Leg Theorem.
4. FB = FA.	4. Corresponding parts.
5. $\triangle$ FAB is isosceles.	5. Definition of isosceles triangle.

16.

1. AE = BF.	1. Given.
2. EF = FE.	2. Identity.
3. AF = BE.	3. Addition of Steps 1 and 2.
4. DF = CE.	4. Given.
5. $\triangle$ AFD and $\triangle$ BEC are right triangles.	5. Given.
6. $\triangle$ AFD $\cong$ $\triangle$ BEC.	6. Hypotenuse-Leg Theorem.
7. $\angle$ AFD $\cong$ $\angle$ BEC.	7. Corresponding parts.
8. $\angle$ x $\cong$ $\angle$ y.	8. Theorem 4-5.

350 17. Area Δ ABC = $\frac{1}{2}sh$.

But by the Pythagorean Theorem, $h = \frac{s}{2}\sqrt{3}$.

Substituting, Area Δ ABC = $\frac{s}{2}(\frac{s}{2}\sqrt{3}) = \frac{s^2}{4}\sqrt{3}$.

351 18.
a. $\sqrt{3}$.
b. $16\sqrt{3}$.
c. $\frac{3}{4}\sqrt{3}$.
d. $\frac{49}{4}\sqrt{3}$.

19. Let s be the length of a side.

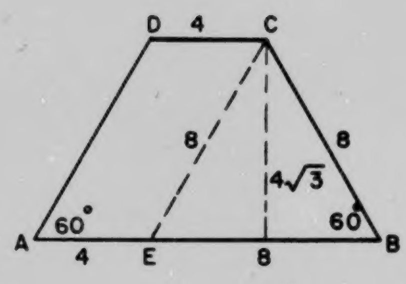
$$\frac{s^2}{4}\sqrt{3} = 9\sqrt{3}.$$
$$s^2 = 4\cdot 9.$$
$$s = 2\cdot 3 = 6.$$
$$h = \frac{s}{2}\sqrt{3} = 3\sqrt{3}.$$

20. Let s be the length of a side.

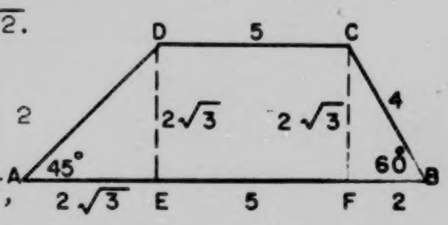
$$\frac{s^2}{4}\sqrt{3} = 16\sqrt{3}.$$
$$s^2 = 4\cdot 16.$$
$$s = 2\cdot 4 = 8.$$
$$h = \frac{s}{2}\sqrt{3} = 4\sqrt{3}.$$

21. A side of the square is 9, and so its perimeter is 36. Then a side of the equilateral triangle is 12. The area of the equilateral triangle equals $36\sqrt{3}$.

22. AC = $9\sqrt{2}$.
AF = $9\sqrt{2}$.
FC = $9\sqrt{2}$.
Therefore Δ FAC is equilateral and m$\angle$ FAC = 60.
Area Δ FAC = $\frac{(9\sqrt{2})^2}{4}\sqrt{3} = \frac{81}{2}\sqrt{3}$.



351 23. Make $\overline{CE} \parallel \overline{DA}$, making equilateral Δ EBC with side of 8. The altitude is $4\sqrt{3}$. Since AB = 12, AE = 4 and DC = 4. Hence, area of trapezoid ABCD $= \frac{1}{2}(4\sqrt{3})(16) = 32\sqrt{2}$.



24. Draw altitudes $\overline{DE}$ and $\overline{CF}$. Since CB = 4, FB = 2 and CF = $2\sqrt{3}$, then DE = $2\sqrt{3}$ and AE = $2\sqrt{3}$, so AB = $7 + 2\sqrt{3}$. Therefore, Area of ABCD $= \frac{1}{2}(2\sqrt{3})(12 + 2\sqrt{3}) = 6 + 12\sqrt{3}$.

352 *25. Since $\overline{CG} \perp$ plane E, then $\overline{CG} \perp \overline{AG}$ and $\overline{CG} \perp \overline{DG}$. $m\angle$ CAG = 45, so Δ CAG is an isosceles right triangle, and CG = AG = 6. Also, AC = $6\sqrt{2}$. In Δ ACD, AC = $6\sqrt{2}$. AD = $2\sqrt{6}$, so by Pythagorean Theorem, DC = $4\sqrt{3}$. In Δ AGD, AG = 6, AD = $2\sqrt{6}$, so DG = $2\sqrt{3}$. Therefore DG = $\frac{1}{2}$DC, so $m\angle$ DCG = 30, and $m\angle$ CDG = 60. Hence, $m\angle$ F-AB-E = 60.

*26. a. In right Δ ADM, DM = $\frac{e}{2}$, so AM = $\frac{\sqrt{3}}{2}e$. In right Δ AMN, AN = $\frac{e}{2}$. By the Theorem of Pythagoras,

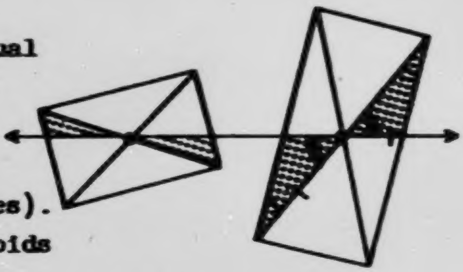
$$(NM)^2 = \left(\frac{e}{2}\sqrt{3}\right)^2 - \left(\frac{e}{2}\right)^2.$$ Hence, NM = $\frac{\sqrt{2}}{2}e$.

b. Δ AHC $\cong$ Δ AHD by Hypotenuse-Leg, and therefore HC = HD. Then H must lie on the perpendicular bisector of $\overline{CD}$. Since in an equilateral triangle the perpendicular bisector the median, and the altitude to any side are the same, H lies on median $\overline{BM}$. Similarly, H must lie on the medians from D and C.

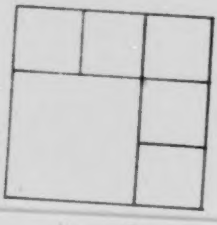
Hence $BH = \frac{2}{3}BM$. But $BM = AM = \frac{\sqrt{3}}{2}e$.
so $BH = \frac{\sqrt{3}}{3}e$. Finally, in ΔABH,
$(AH)^2 = (AB)^2 - (BH)^2 = e^2 - \left(\frac{\sqrt{3}}{3}e\right)^2 = \frac{2}{3}e^2$.
Hence, $AH = \frac{\sqrt{6}}{3}e$.

353 27. $\overline{YA} \perp \overline{AB}$ and $\overline{DA} \perp \overline{AB}$ because of the given square and rectangle. By definition $\angle YAD$ is the plane angle of $\angle X\text{-}AB\text{-}E$ and hence $m\angle YAD = 60$. By definition of projection $\overline{YD} \perp E$ and hence $m\angle ADY = 90$. Then $m\angle AYD = 30$ and $AD = \frac{1}{2}AY$. Therefore area ABCD $= \frac{1}{2}$ area ABXY = 18.

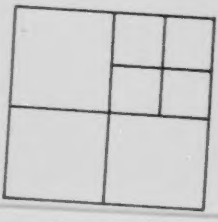
*28. Find the point of intersection of the diagonals of each rectangle. A line containing these intersection points separates each rectangle into two trapezoidal regions of equal area (or in special cases the line may contain a diagonal and the regions will be congruent triangles). The proof that the trapezoids are equal in area involves showing the pairs of shaded triangles congruent by A.S.A.



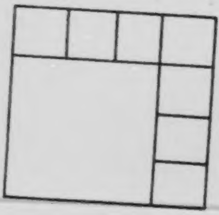
353 Here is a problem that might be interesting to the class. It has to do with cutting up a square into a certain number of smaller squares, not necessarily equal in area. We will talk of an integer k, as being "acceptable" if a square can be subdivided into k squares. For example, given any square we can divide it into 4 squares, but not into 2, 3, or 5. Try it. Below are some diagrams showing how a square may be divided into 6, 7, and 8 smaller squares:



k = 6



k = 7

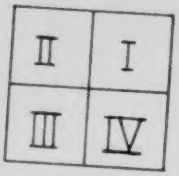


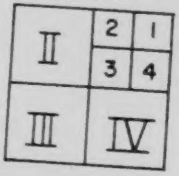
k = 8

We may ask is there some pattern or some integer k, above which this will always be possible. Actually any $k \geq 6$ will always be acceptable.

We now show that if a square can be divided into k smaller squares, then it can be divided into k + 3 smaller squares: Imagine that we have already divided a square into k squares. Now, split one of the squares into 4 smaller squares by bisecting the sides. In this process we have lost one larger square and gained four smaller ones, thus gaining three.

We illustrate using k = 4:





After dividing the original square into 4 smaller squares, we take one of them, and divide it into 4 squares. Instead of having 4 squares from the first division we have only three, and now have 4 additional ones giving a total of 7. Since we know that $k = 6$, $k = 7$, and $k = 8$ are acceptable, and that we can get $k + 3$ squares from any division, we can form the following sequences:

$$6,\quad 9,\quad 12,\quad 15,\quad \ldots,$$
$$7,\quad 10,\quad 13,\quad 16,\quad \ldots,$$
$$8,\quad 11,\quad 14,\quad 17,\quad \ldots,$$

Hence all $k \geq 6$ are acceptable.

Review Problems

353 1. four.

2. 12. This may be found by first showing that the area of the triangle is 36.

354 3. 10 miles.

4. $\frac{15\sqrt{2}}{2}$.

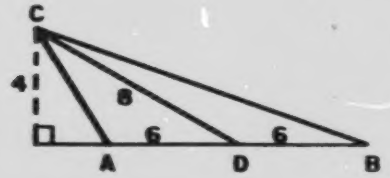
5. 48.

6. a. 35. b. 5.

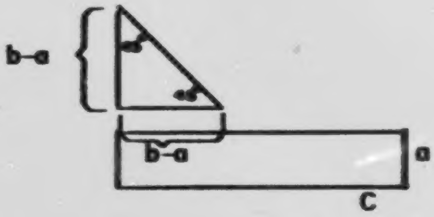
7. Let the length of the side of the triangle be $2n$. Then $(2n)^2 = n^2 + 6^2$ and $n = 2\sqrt{3}$, so $2n = 4\sqrt{3}$.

8. The diagonals of a rhombus are perpendicular and bisect each other, forming four congruent right triangles. By the Pythagorean Theorem, half the length of the other diagonal is 5. Each triangle has an area of 30. The area of the rhombus is 120.

354 9. 24.



10. Separate the figure into a rectangular and a triangular region. The area of the rectangle is ac. The area of the triangle is $\frac{1}{2}(b - a)^2$. The total area is $ac + \frac{1}{2}(b - a)^2$.

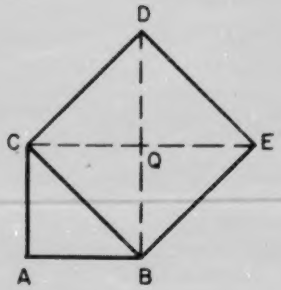


355 11. The outer triangle has an area of $\frac{1}{2}bc$. The inner triangle has an area $\frac{1}{2}(b - 3a)(c - 4a)$. The area of the shaded portion is found by subtraction to be $\frac{1}{2}(3ac + 4ab - 12a^2)$.

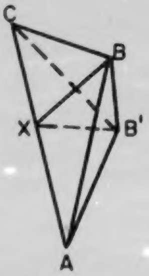
12. 1010.

13. Consider $\overline{BX}$ as a base for Δ BXC and $\overline{BA}$ as a base for parallelogram ADCB. Then area Δ BXC = $\frac{1}{4}$ area parallelogram ADCB. By a similar argument, area Δ CXD = $\frac{1}{4}$ area parallelogram ADCB. Subtracting the areas of these two triangles from that of the parallelogram we find that area ABCX = $\frac{1}{2}$ area parallelogram ABCD.

14. Let the length of the side of the isosceles right triangle be e. Then its hypotenuse has length $e\sqrt{2}$, and the area of a square on the hypotenuse is $(e\sqrt{2})^2 = 2e^2$. The area of the triangle is $\frac{1}{2}e^2$, which is one-fourth that of the square.



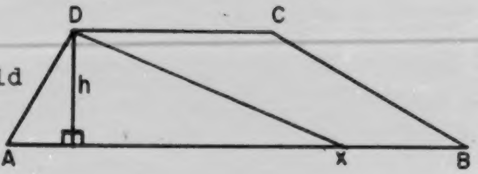
Alternate solution: The five triangles in the drawing are all congruent, so by Postulate 18 all have the same area. Therefore, by Postulate 19, area BCDE = 4 area Δ ABC.



355 *15. Let ABC be the given triangle and AB'C its projection on the plane. Let X be the mid-point of $\overline{AC}$, the side lying in the plane.

1. $\overline{BB'} \perp \overline{B'C}$, $\overline{BB'} \perp \overline{B'A}$ and $\overline{BB'} \perp \overline{B'X}$.	1. Definition of projection. Definition of a line perpendicular to a plane.
2. Δ AB'B $\cong$ Δ CB'B.	2. Hypotenuse-Leg Theorem.
3. CB' = AB' and Δ AB'C is isosceles.	3. Corresponding parts and Definition of isosceles.
4. $\overline{BX}$ is an altitude of Δ ABC; $\overline{B'X}$ is an altitude of Δ AB'C.	4. The median to the base of an isosceles triangle is an altitude.
5. $m\angle$ BXB' = 60.	5. Given, and Definition of plane angle of a dihedral angle.
6. $m\angle$ XBB' = 30.	6. Corollary 9-13-2.
7. B'X = $\frac{1}{2}$BX.	7. 30-60 Triangle Theorem.
8. Area Δ AB'C = $\frac{1}{2}$ Area Δ ABC.	8. Theorem 11-2.

356 *16. On $\overline{AB}$, the longer of the two parallel sides, locate a point X so that $AX = \frac{1}{2}(AB + CD)$. Then $\overline{DX}$ separates the trapezoid into two regions of equal area.



Proof: Area $\Delta ADX = \frac{1}{2}h(AX)$.
Area $XBCD = \frac{1}{2}h(XB + CD)$.
For these areas to be equal it is necessary that

$$\frac{1}{2}h(AX) = \frac{1}{2}h(XB + CD),$$ which will be the case if

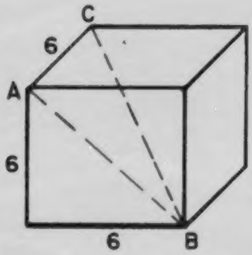
$$AX = XB + CD.$$

Since $XB = AB - AX$, the previous equation can be written

$$AX = AB - AX + CD,$$ from which

$$AX = \frac{1}{2}(AB + CD).$$

*17. By the Pythagorean Theorem any face diagonal such as $\overline{AB}$ has length $\sqrt{72}$. The diagonal $\overline{CB}$ has length $\sqrt{36 + 72} = \sqrt{108}$ or $6\sqrt{3}$.

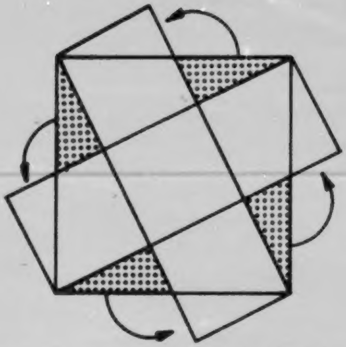


*18. $AC = \sqrt{200} = 10\sqrt{2}$.
$AG = 15$.

*19. $BE = 12$.

1. $\Delta CFD \cong \Delta CEB$.	1. A.S.A.
2. $CF = CE$.	2. Corresponding parts.
3. $(BC)^2 = 256$, or $BC = 16$.	3. Given area of the square.
4. $\frac{1}{2}(CE)(CF) = \frac{1}{2}(CE)^2$ $= 200$, or $CE = 20$.	4. Given and Statement 2.
5. $BE = 12$.	5. Pythagorean Theorem.

356 *20. The area of RSPQ is $\frac{1}{5}$ that of ABCD as can be seen by rearranging the triangular regions as shown.



357 *21. b. There are 45 small squares and 10 half squares so the area is 50 square units.

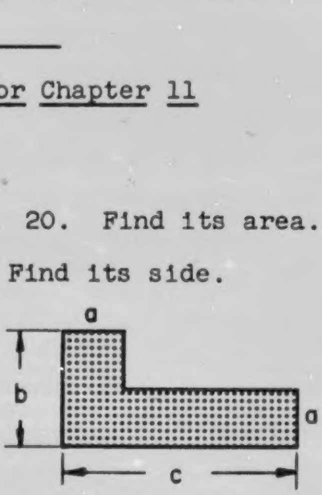
c. There are 42 small squares and 14 half squares so the area is 49 square units.

d. The area of the first triangle is $\frac{1}{2}\cdot 10\cdot 10 = 50$;
The area of the second triangle is $\frac{1}{2}\cdot 14\cdot 7 = 49$.
A leg of the first is 10, and a leg of the second is $7\sqrt{2}$ or approximately 9.90. One-tenth unit in length is too small to notice when cutting one triangle out and placing it on the other.

Illustrative Test Items for Chapter 11

A. Area Formulas.

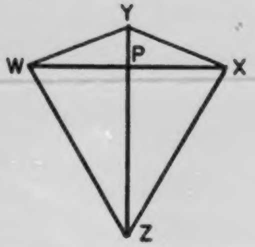
1. The perimeter of a square is 20. Find its area.
2. The area of a square is n. Find its side.
3. Find the area of the figure in terms of the lengths indicated.



4. The base of a rectangle is three times as long as the altitude. The area is 147 square inches. Find the base and the altitude.

5. The area of a triangle is 72. If one side is 12, what is the altitude to that side?

6. In the figure $WY = XY$ and $WZ = XZ$. $WX = 8$ and $YZ = 12$. Find the area of WZXY.

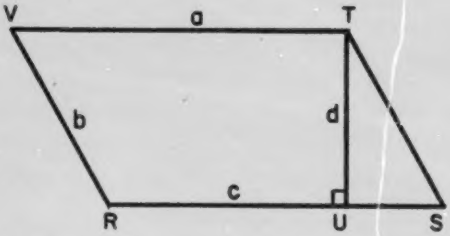


7. RSTV is a parallelogram. If the small letters in the drawing represent lengths, give the area of:

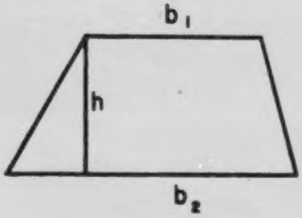
   a. Parallelogram RSTV.

   b. Δ STU.

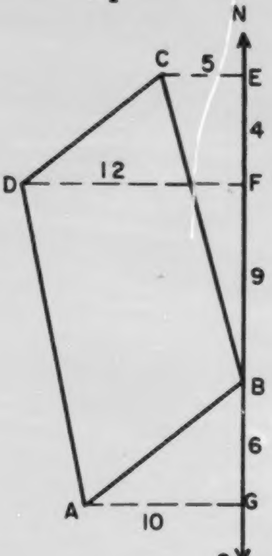
   c. Quadrilateral VRUT.



8. Show how a formula for the area of a trapezoid may be obtained from the formula $A = \frac{1}{2}bh$ for the area of a triangle.

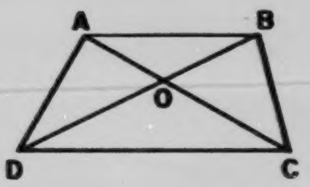


9. In surveying field ABCD shown here a surveyor laid off north and south line $\overleftrightarrow{NS}$ through B and then located the east and west lines $\overleftrightarrow{CE}$, $\overleftrightarrow{DF}$ and $\overleftrightarrow{AG}$. He found that $CE = 5$ rods, $AG = 10$ rods, $BG = 6$ rods, $BF = 9$ rods and $FE = 4$ rods. Find the area of the field.

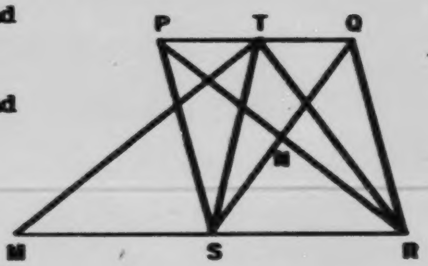


B. Comparison of Areas.

1. Given: ABCD is a trapezoid. Diagonals $\overline{AC}$ and $\overline{BD}$ intersect at O.
   Prove: Area Δ AOD = Area Δ BOC.



2. In this figure PQRS is a parallelogram with PT = TQ and MS = SR. In a through e below compare the areas of the two figures listed.
   - a. Parallelogram SRQP and Δ SQR.
   - b. Parallelogram SRQP and Δ MTR.
   - c. Δ PMS and Δ MTR.
   - d. Δ STR and Δ SPR.
   - e. Δ MTR and Δ RQT.

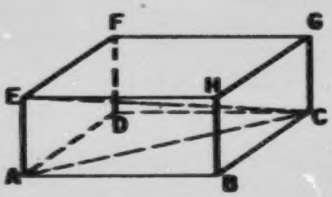


C. Pythagorean Theorem.

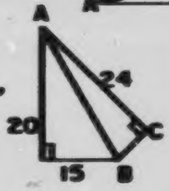
1. How long must a tent rope be to reach from the top of a 12 foot pole to a point on the ground which is 16 feet from the foot of the pole?

2. A boat travels south 24 miles, then east 6 miles, and then north 16 miles. How far is it from its starting point?

3. Given the rectangular solid at the right with AB = 12, BC = 16 and BH = 15. Find AC and EC.



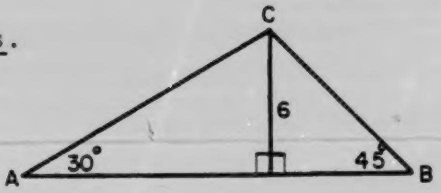
4. For the figure at the right, find AB and CB.



D. Properties of Special Triangles.

1. a. What is the length of $\overline{CB}$?

   b. What is the length of $\overline{AC}$?



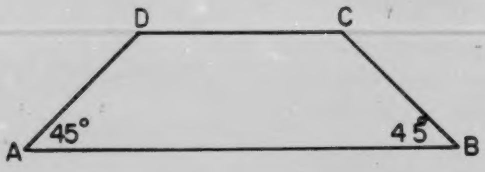
2. The diagonal of a square is $\sqrt{2}$. Find its side.

3. The longest and shortest sides of a right triangle are 10 and 20. What is the measure of the smallest angle of the triangle?

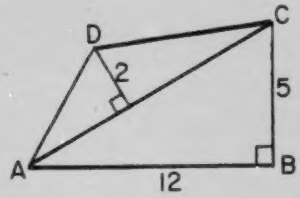
4. The measures of each of two angles of a triangle is 45. What is the ratio of the longest side to either of the other sides?

E. Miscellaneous Problems.

1. ABCD is a trapezoid. CD = 1 and AB = 5. What is the area of the trapezoid?



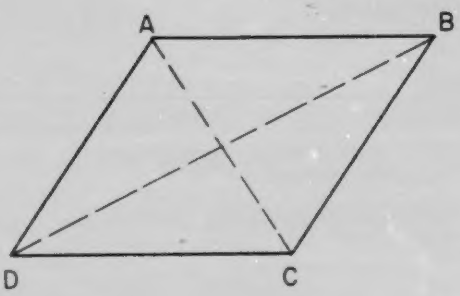
2. What is the area of ABCD?



3. ABCD is a rhombus with AC = 24 and AB = 20.

   a. Compute its area.

   b. Compute the length of the altitude to $\overline{DC}$.

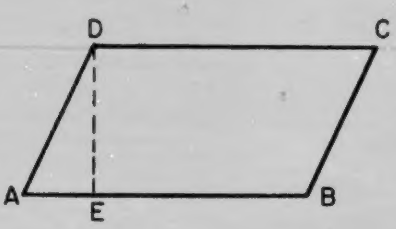


4. Find the area of a triangle whose sides are 9", 12", and 15".

5. ABCD is a parallelogram with altitude $\overline{DE}$. Find the area of the parallelogram if:

   a. $AB = 2\frac{1}{2}$ and $DE = 6\frac{1}{3}$.

   b. $AB = 10$, $AD = 4$, and $m\angle A = 30$.



6. Find the area of an isosceles triangle which has congruent sides of length 8 and base angles of 30°.

Answers

A. 1. 25.

2. $\sqrt{n}$.

3. $ab + a(c - a)$, or $ac + a(b - a)$, or $ab + ac - a^2$.

4. Let a be the length of the altitude and $3a$ the length of the base. Then

$$3a^2 = 147$$
$$a^2 = 49$$
$$a = 7.$$

The altitude is 7. The length of the base is 21.

5. 12.

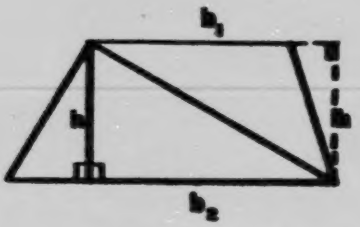
6. Consider the figure to be the union of triangular regions WYZ and XYZ. It can be proved that $\overline{YZ}$ is the perpendicular bisector of $\overline{WX}$. Hence $\overline{WP}$ and $\overline{XP}$ are altitudes of triangle WYZ and XYZ respectively. The area of each of these triangles is 24. Hence the area of WZXY is 48.

7. a. ad.

   b. $\frac{1}{2}d(a - c)$.

   c. $\frac{1}{2}d(a + c)$.

8. Separate the figure into triangular regions by drawing a diagonal. The areas of the respective triangles are $\frac{1}{2}b_1h$ and $\frac{1}{2}b_2h$. The sum of these two areas is $\frac{1}{2}b_1h + \frac{1}{2}b_2h = \frac{1}{2}h(b_1 + b_2)$.



9. Area ABCD = Area AGFD + Area DFEC - Area AGB - Area CEB.

   Area ABCD = 165 + 34 - 30 - $32\frac{1}{2}$.

   Area ABCD = $136\frac{1}{2}$.

   The area of the field is $136\frac{1}{2}$ square rods.

B. 1. Area Δ ADC = Area Δ BCD because the triangles have the same base $\overline{DC}$ and equal altitudes.
   Area Δ DOC = Area Δ DOC.
   Therefore, by subtracting, we have Area Δ AOD
   = Area Δ BOC.

   2. a. Area parallelogram SRQP
      = 2 Area Δ SQR.

      b. Area parallelogram SRQP
      = Area Δ MTR.

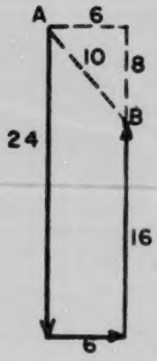
      c. Area Δ PMS = $\frac{1}{4}$ Area Δ MTR.

      d. Area Δ STR = Area Δ SPR.

      e. Area Δ MTR = 4 Area Δ RQT.

C. 1. 20 feet.

2. 10 miles.
(see figure at right.)


A
6
10
8
B
24
16
6

3. AC = 20.
EC = 25.

4. AB = 25 and CB = 7.

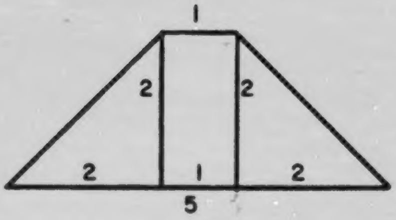
D. 1. a. $6\sqrt{2}$. b. 12.

2. 1.

3. 30.

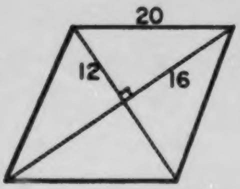
4. $\sqrt{2}$ to 1.

E. 1. 6. (see figure at right.)



2. 43. (AC = 13.)

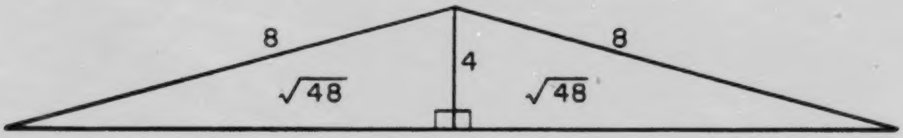
3. a. 384. (See figure at right)
b. 19.2 (384 ÷ 20.)



4. 54. ($\frac{1}{2}\cdot 9\cdot 12$. The triangle is a right triangle.)

5. a. $15\frac{5}{6}$. b. 20.

6. $16\sqrt{3}$. (From $4\sqrt{48}$.)


8
8
4
$\sqrt{48}$
$\sqrt{48}$

Chapter 12

SIMILARITY

In Chapter 5 we explored the concept of congruence, which encompassed the idea of a one-to-one correspondence between the vertices of two triangles such that corresponding sides and corresponding angles were congruent. In this chapter we talk of a correspondence between triangles such that corresponding angles are congruent and the ratios of corresponding sides are equal. This correspondence is called a similarity. After a discussion of proportions, there appears a proof of the fundamental proportionality theorem for triangles that is different from the usual one given. This proof is not new; quite the contrary. It was found in a text-book, published in 1855, written by the noted French mathematician, A. M. Legendre. More will be said about it later. For the most part, this chapter presents a conventional treatment of similar triangles.

360 The student is expected to call upon his algebra in working with proportionalities. We should need no statements about the algebraic properties of proportions. The four properties we do state, however, will provide a basis for practice and review. The quantities used in proportions are numbers, and the algebra of fractional equations will enable the student to do all that is required.

361 The geometric mean of two positive numbers, a and c, is the positive number b, such that $\frac{a}{b} = \frac{b}{c}$. You may recognize that b is what has been called, in some text-books, the mean proportional between a and c. We speak of this as the geometric mean of a and c, and $b = \sqrt{ac}$. Then "geometric mean" and "mean proportional" are names for the same thing, and we prefer to use "geometric mean" in this text. In mathematics there are such things as harmonic and arithmetic means that do not arise from proportions, and we have used "geometric mean" because it arose historically in a geometric construction.

Problem Set 12-1

361 1. a. $7a = 3b$. b. $4x = 3$. c. $6y = 20$.

362 2. a. $\frac{3}{2}$. c. $\frac{65}{4}$.

b. $\frac{35}{4}$. d. $\frac{33}{2}$.

3. a. $\frac{a}{x} = \frac{2}{3}$ and $\frac{a}{2} = \frac{x}{3}$.

b. $\frac{4}{3} = \frac{5}{m}$ and $\frac{m}{3} = \frac{5}{4}$.

c. $\frac{a}{b} = \frac{7}{4}$ and $\frac{b}{a} = \frac{4}{7}$.

d. $\frac{x}{5} = \frac{9}{6}$ and $\frac{5}{x} = \frac{6}{9}$.

4. a. $a = \frac{6bc}{5d}$. c. $a = \frac{21bd}{20c}$.

b. $a = \frac{22bd}{35c}$. d. $a = \frac{12cd}{5b}$.

*5. a. $\frac{a + b}{b} = \frac{4}{1}$ and $\frac{a - b}{b} = \frac{2}{1}$.

b. $\frac{y + 2}{2} = \frac{x + 3}{3}$ and $\frac{y - 2}{2} = \frac{x - 3}{3}$.

c. $\frac{a}{c} = \frac{4}{7}$ and $\frac{a - c}{c} = \frac{-3}{7}$.

d. $\frac{b + a}{a} = \frac{8}{5}$ and $\frac{b - a}{a} = \frac{-2}{5}$.

363 6. a. $1, \frac{7}{3}, 4$.

b. $1, \frac{7}{3}, 4$.

c. $1, \frac{7}{3}, 4$.

The three new sequences are identical, so each pair of the original three sequences are proportional.

363 7. a and d.
a and i.
d and i.
b and f.
b and h.
f and h.
c and e.
c and g.
e and g.

a. $1, \frac{7}{5}, \frac{9}{5}$.
b. 1, 2, 3.
c. $1, \frac{7}{9}, \frac{17}{9}$.
d. $1, \frac{7}{5}, \frac{9}{5}$.
e. $1, \frac{7}{9}, \frac{17}{9}$.
f. 1, 2, 3.
g. $1, \frac{7}{9}, \frac{17}{9}$.
h. 1, 2, 3.
i. $1, \frac{7}{5}, \frac{9}{5}$.

8. $w = 800$; $v = 1000$.

9. $x = \frac{3}{4}$; $y = 1$; $z = \frac{11}{4}$.

10. b and f are correct.

364 11. $p = 18$; $q = 24$; $t = 70$.

12. a. G.M. = 6, (6.000); A.M. = 6.5.
b. G.M. = $6\sqrt{2}$, (8.484); A.M. = 9.0.
c. G.M. = $4\sqrt{5}$, (8.944); A.M. = 9.0.
d. G.M. = $4\sqrt{3}$, (6.928); A.M. = 13.0.
e. G.M. = $\sqrt{6}$, (2.449); A.M. = 2.5.

---

364 The definition of a similarity, like the definition of a congruence, requires two things. For similar triangles we could have based our definition on either one of the two conditions, and proved the other. It seems best, however, to make a definition which may be generalized for other polygonal figures.

365 Notice that the idea of a correspondence which matches vertices is employed for similar triangles as for congruent triangles: the similarity indicates, without recourse to a figure, the corresponding sides and angles.

Problem Set 12-2

366 1. a. $AB = \frac{AC \cdot DE}{DF}$. d. $AB = \frac{DE \cdot BC}{EF}$.

b. $BC = \frac{AB \cdot EF}{DE}$. e. $BC = \frac{AC \cdot EF}{DF}$.

c. $AC = \frac{BC \cdot DF}{EF}$. f. $AC = \frac{DF \cdot AB}{DE}$.

367 2. a, b; $\frac{3}{6} = \frac{4}{8} = \frac{6}{12}$.

a, d; $\frac{3}{9} = \frac{4}{12} = \frac{6}{18}$.

b, d; $\frac{8}{12} = \frac{6}{9} = \frac{12}{18}$.

3. $\frac{2}{7.5} = \frac{1.6}{h}$.

$h = \frac{(7.5)(1.6)}{2}$.

$h = 6$.

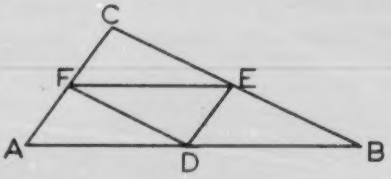
The height of the object in the enlargement is 6 inches.

4. Yes. If $\Delta ABC \cong \Delta A'B'C'$, the conditions necessary for a similarity are met. That is,

(1) $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, $\angle C \cong \angle C'$ and

(2) $\frac{A'B'}{AB} = \frac{A'C'}{AC} = \frac{B'C'}{BC}$.

367 5. Given: Δ ABC; D, E, F the mid-points of the sides $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ respectively.
Prove: Δ EFD $\sim$ Δ ABC.
Proof: By Theorem 9-22, $ED = \frac{1}{2}AC$, $FE = \frac{1}{2}AB$, $FD = \frac{1}{2}CB$, and $\overline{ED} \parallel \overline{AC}$, $\overline{FE} \parallel \overline{AB}$, $\overline{FD} \parallel \overline{CB}$. FDEC, ADEF, DBEF are parallelograms. By Theorem 9-16, $\angle$ FDE $\cong$ $\angle$ BCA, $\angle$ DEF $\cong$ $\angle$ CAB, $\angle$ EFD $\cong$ $\angle$ ABC; since we have also proved above that $\frac{ED}{AC} = \frac{FE}{AB} = \frac{FD}{CB}$, Δ EFD $\sim$ Δ ABC by definition of similarity.



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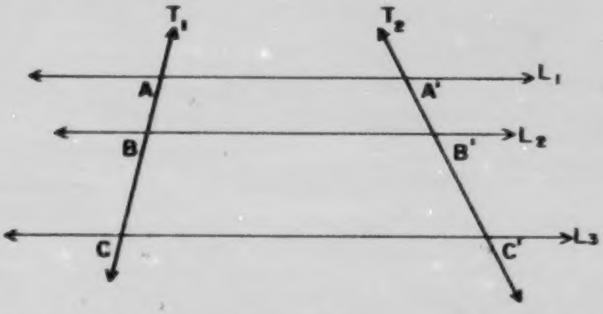
368 Conventional proofs of the Basic Proportionality Theorem contended with (1) a relatively unconvincing division of the sides of a triangle by a series of parallel lines, and (2) the problem of what to do when the ratio of the length of a segment to the length of a side containing that segment is not a rational number (the incommensurable case). It has often been the practice to give a proof of the theorem for the commensurable case and mention the other possibility. The proof in the text avoids this difficulty since it is based on the area postulates, which involve real numbers.

369 In the proof of Theorem 12-2 we tacitly assume that E is between A and C'. It is obvious from a figure that betweenness is preserved under parallel projection, but we have not justified it on the basis of our postulates. It is easily proved as follows:

(The Parallel Projection Theorem.)

Given two transversals T_1 and T_2 intersecting three parallel lines L_1, L_2, L_3 in points A, B, and C and A', B', and C' respectively. If B is between A and C then B' is between A' and C'.



Proof: Since $L_1 \parallel L_2$, then the segment $\overline{AA'}$ cannot intersect L_2 and hence A and A' are on the same side of L_2. Likewise, since $L_3 \parallel L_2$, then the segment $\overline{CC'}$ cannot intersect L_2 and C and C' are on the same side of L_2. Since B is between A and C by hypothesis, segment $\overline{AC}$ intersects L_2 at B; hence, A and C are on opposite sides of L_2. Since A' and A are in the same half-plane determined by L_2 and C' and C are in the same half-plane and A and C are in opposite half-planes then it follows that A' and C' are in opposite half-planes determined by L_2. Hence $\overline{A'C'}$ meets L_2 in a point which must be B', since B' is the intersection of $\overleftrightarrow{A'C'}$ and L_2. Therefore, B' is between A' and C'.
370 We have assumed that $A \neq A'$ and $C \neq C'$. The argument above is easily modified to apply to the cases where $A = A'$ or $C = C'$.

Note that the application of this principle to Theorem 12-2 involves the case $A = A'$.

Problem Set 12-3a

370 1. $\frac{a+b}{a} = \frac{x+y}{x}$. $\frac{a}{b} = \frac{x}{y}$.

$\frac{a+b}{b} = \frac{x+y}{y}$. $\frac{a}{x} = \frac{b}{y}$.

$\frac{a+b}{x+y} = \frac{a}{x}$. $\frac{x+y}{a+b} = \frac{y}{b}$.

2. $\frac{FA}{FH} = \frac{FB}{FT}$. $\frac{TB}{FT} = \frac{HA}{FH}$.

$\frac{FA}{HA} = \frac{FB}{TB}$. $\frac{FT}{FH} = \frac{FB}{FA} = \frac{TB}{HA}$.

$\frac{FH}{HA} = \frac{FT}{TB}$. $\frac{BT}{AH} = \frac{BF}{AF} = \frac{FT}{FH}$.

3. a. $AB = 5\frac{5}{7}$. b. $BF = 5$. c. $BF = 13\frac{1}{2}$.

371 4. a. $BC = 24$. d. $BE = 7\frac{1}{2}$.

b. $CE = 6\frac{2}{3}$. e. $AD = 10$.

c. $AC = 11$.

5. No. $\frac{20}{16} \neq \frac{30}{23}$.

6. a, b, e.

372 7. a. By Theorem 12-1, $\frac{CA}{CD} = \frac{CB}{CF}$.

Then

$$\frac{CA}{CD} - 1 = \frac{CB}{CF} - 1.$$

or

$$\frac{CA - CD}{CD} = \frac{CB - CF}{CF}.$$

Therefore

$$\frac{DA}{CD} = \frac{FB}{CF}.$$

372 b. Taking the reciprocals of both fractions of (a) we get

$$\frac{CD}{DA} = \frac{CF}{FB}.$$

Then

$$\frac{CD}{DA} + 1 = \frac{CF}{FB} + 1$$

or

$$\frac{CD + DA}{DA} = \frac{CF + FB}{FB}.$$

Therefore,

$$\frac{CA}{DA} = \frac{CB}{FB}.$$

c. By Theorem 12-1, $\frac{CA}{CD} = \frac{CB}{CF}$.

Clearing of fractions, $CA \cdot CF = CD \cdot CB$, and dividing by $CF \cdot CB$ we have

$$\frac{CA}{CB} = \frac{CD}{CF}.$$

8. $\frac{w}{19} = \frac{9}{16}$ is one. $w = \frac{171}{16}$.

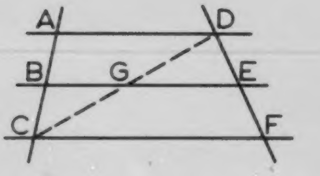
9. x must be 8 or 11.

10.

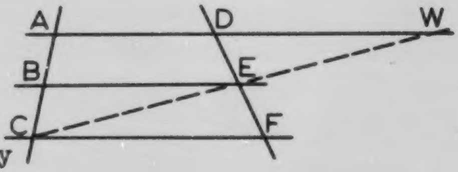
1. $\overline{EF} \parallel \overline{AB}$. $\overline{FG} \parallel \overline{BC}$. $\overline{GH} \parallel \overline{DC}$.	1. Given.
2. $\frac{XA}{XE} = \frac{XB}{XF}$. $\frac{XB}{XF} = \frac{XC}{XG}$. $\frac{XC}{XG} = \frac{XD}{XH}$.	2. Theorem 12-1.
3. $\frac{XA}{XE} = \frac{XD}{XH}$.	3. From Step 2.
4. $\overline{HE} \parallel \overline{AD}$.	4. Theorem 12-2.

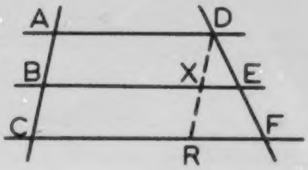
No, the figure does not have to be planar.

373 11. Proof: Draw transversal $\overleftrightarrow{DC}$ intersecting $\overleftrightarrow{BE}$ in G. In Δ CAD we have by Theorem 12-1, $\frac{AC}{BC} = \frac{CD}{CG}$ from which $\frac{AB}{BC} = \frac{DG}{GC}$. Similarly, in Δ DCF, we get $\frac{DG}{GC} = \frac{DE}{EF}$. Hence, $\frac{AB}{BC} = \frac{DE}{EF}$.



(An alternate method of proof might use an auxiliary line $\overleftrightarrow{CW}$ as shown at the right, or a line $\overleftrightarrow{DR} \parallel \overleftrightarrow{AC}$ as shown here.)





12. Lot I: 80 feet. Lot II: 160 feet. Lot III: 120 feet.

13. Since $\overleftrightarrow{AB} \parallel \overleftrightarrow{XY}$, $\frac{OA}{OX} = \frac{OB}{OY}$.

Similarly, $\overleftrightarrow{BC} \parallel \overleftrightarrow{YZ}$ implies

$$\frac{OB}{OY} = \frac{OC}{OZ}.$$

Hence, $\frac{OA}{OX} = \frac{OC}{OZ}$. This implies $\overleftrightarrow{AC} \parallel \overleftrightarrow{XZ}$ by Theorem 12-2.

374 14. x will be the length of the folded card, so

$$\frac{6}{x} = \frac{x}{3} \text{ and } x^2 = 18.$$

The width of the card should be $\sqrt{18}$ or $3\sqrt{2}$ inches.

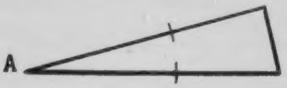
374-378 In the proofs of Theorems 12-3, 12-4, and 12-5 we have drawn the figure with $AB > DE$ and used this in each proof, except that in Theorem 12-3 the case $AB = DE$ was discussed. (Notice here if $AB = DE$, $\triangle AE'F'$ and $\triangle ABC$ coincide, that is $\triangle AE'F' = \triangle ABC$.) In the case $AB < DE$ a similar proof would be given with E' on $\overrightarrow{DE}$ and $DE' = AB$.

It might be advisable to point out to the students the general plan of the proof of Theorem 12-5. First prove $\triangle ABC \sim \triangle AE'F'$ by the A.A. Corollary, then prove $\triangle AE'F' \cong \triangle DEF$ by the S.S.S. Theorem, and finally prove $\triangle ABC \sim \triangle DEF$ by the A.A. Corollary.

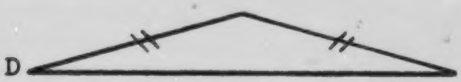
Problem Set 12-3b

379 1. Similarities are indicated in a, c, d.

Notice that the wording of (e) permits



and



2. The A.A.A. and the A.A. Theorems.

3. a. No. c. No.

b. Yes. d. Yes.

380 4. a. The triangles are similar. S.S.S.

b. Not similar.

c. The triangles are similar. A.A.A. or S.A.S.

d. Similar. A.A.A.

e. Similar. S.S.S.

f. Similar. A.A. or S.A.S.

380 5. a. $\angle AXC$ or $\angle BXC$.

b. $\angle ACX$.

c. $\triangle AXC$, or $\triangle CXB$.

6. $XC = \frac{qr}{p}$, or $XG = \frac{qp}{r}$. No.

381 7. a. $\triangle ABP \sim \triangle QRS$. $\frac{AB}{QR} = \frac{AP}{QS} = \frac{BP}{RS} = \frac{1}{3}$.

b. $\triangle MTW \sim \triangle RLS$. $\frac{MT}{RL} = \frac{MW}{RS} = \frac{TW}{LS} = \frac{2}{3}$.

c. $\triangle ABC$ is not $\sim \triangle XYZ$.

d. $\triangle ABC \sim \triangle TSR$. $\frac{AB}{TS} = \frac{AC}{TR} = \frac{BC}{SR} = \frac{1}{5}$.

e. $\triangle ABC \sim \triangle TWX$. $\frac{AB}{TW} = \frac{BC}{WX} = \frac{AC}{TX} = 6$.

8. $\triangle ABC \sim \triangle CDL$ since the vertical angles at L are congruent as well as the given angles B and D. From the given information $\frac{CD}{AB} = \frac{4}{1}$. Since the triangles have been proved similar $\frac{DL}{BL} = \frac{4}{1}$. Then $\frac{DL + BL}{BL} = \frac{4 + 1}{1}$. Since L is between B and D, this can be written $\frac{BD}{BL} = \frac{5}{1}$ or $BD = 5BL$.

382 9. a. $\frac{r}{1} = \frac{s}{x}$, $rx = s$, $x = \frac{s}{r}$.

b. $\frac{1}{m} = \frac{p}{x}$, $x = mp$.

c. $\frac{1}{k} = \frac{k}{x}$, $x = k^2$.

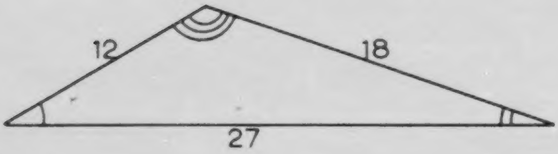
d. $\frac{t}{1} = \frac{1}{x}$, $xt = 1$, $x = \frac{1}{t}$.

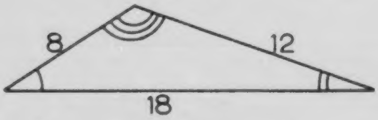
e. Part b.

f. Part a.

g. No.

382 10. Of the five equal pairs of parts three must be angles, for if three were sides the triangles would be congruent. Hence the triangles are similar. Neither of the two pairs of equal sides can be corresponding sides or the triangles would be congruent by A.S.A. The remaining possibility can best be shown by an example.





11. 1. $\Delta OBX \sim \Delta O_1B_1X$ by A.A.A.

2. Therefore $\frac{OB}{O_1B_1} = \frac{OX}{O_1X}$.

3. $\Delta ODX \sim \Delta O_1D_1X$ by A.A.A.

4. Therefore $\frac{OD}{O_1D_1} = \frac{OX}{O_1X}$.

5. From Statements 2 and 3, $\frac{OB}{O_1B_1} = \frac{OD}{O_1D_1}$.

*12. a. $\Delta BSC \sim \Delta BTD$, $\Delta DSC \sim \Delta DRB$, $\Delta RSB \sim \Delta DST$.

b. $\frac{z}{y} = \frac{p}{p + q}$.

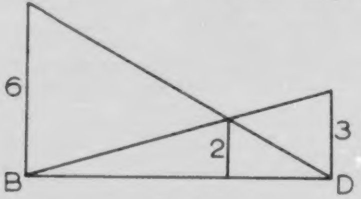
c. $\frac{z}{x} = \frac{q}{p + q}$.

d. $\frac{z}{x} + \frac{z}{y} = \frac{p + q}{p + q}$.

$\frac{z}{x} + \frac{z}{y} = 1$.

$\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$.

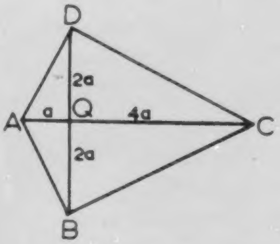
383 e. Construct perpendiculars 6 and 3 units long at opposite ends (but on the same side) of any segment $\overline{BD}$. Join the ends of these perpendiculars to the opposite ends of the segment, and where these lines intersect, draw a perpendicular to $\overline{BD}$. Measure this perpendicular. It should be 2 units long. Therefore the task would require 2 hours.



13.

1.	ABRQ is a parallelogram.	1.	Given.
2.	$\angle QHA \cong \angle BHF$.	2.	Vertical angles.
3.	$\overline{AQ} \parallel \overline{RB}$.	3.	Definition of a parallelogram.
4.	$\angle AQB \cong \angle FBH$.	4.	Alternate interior angles.
5.	$\Delta AHQ \sim \Delta FHB$.	5.	A.A.
6.	$\frac{AH}{FH} = \frac{HQ}{HB}$.	6.	Definition of similar triangles.
7.	$AH \cdot HB = FH \cdot HQ$.	7.	Clearing of fractions.

14. a. and b. Let a, 2a, 4a stand for the lengths as shown in the figure. Then it can easily be shown for each pair of triangles mentioned that the S.A.S. Similarity Theorem applies.

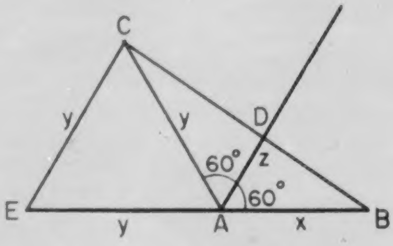


383 c. $\angle ADQ$ and $\angle QAD$ are complementary angles. $\angle QAD \cong \angle QDC$, since they are corresponding angles of similar triangles. Therefore $\angle ADQ$ and $\angle QDC$ are complementary and $m\angle ADC = 90$.

15. Let $\overleftrightarrow{BE}$ be parallel to $\overleftrightarrow{AD}$, meeting $\overleftrightarrow{AC}$ in E. $\angle ABE \cong \angle DAB$ (alt. int. $\angle$s) and $\angle AEB \cong \angle CAD$ (corr. $\angle$s). Also, $\angle DAB \cong \angle CAD$ (given). Therefore $\angle AEB \cong \angle ABE$. Therefore $AE = AB$. Since $\frac{CD}{DB} = \frac{CA}{AE}$, then $\frac{CD}{DB} = \frac{CA}{AB}$ by substitution.

384 *16. From the previous problem $\frac{CD}{DB} = \frac{CA}{AB}$. By an exactly similar proof you can show that $\frac{CD'}{D'B} = \frac{CA}{AB}$. Therefore $\frac{CD'}{D'B} = \frac{CD}{DB}$.

*17. a. Let E be the point on the ray opposite to $\overrightarrow{AB}$ such that $AE = y$. Then ΔAEC is equilateral, $EC = y$. In the similar triangles ECB and ADB,



$$\frac{EC}{AD} = \frac{EB}{AB}, \text{ or}$$

$$\frac{y}{z} = \frac{x + y}{x},$$

$$\frac{y}{z} = 1 + \frac{y}{x}.$$

Dividing by y, we get

$$\frac{1}{z} = \frac{1}{y} + \frac{1}{x}.$$

b. Yes, place the straight-edge against R_1 on the middle scale and R_2 on one of the outer scales. Then read off R on the other outer scale.

385 18.

1. $\frac{RW}{AL} = \frac{WS}{LQ} = \frac{RT}{AM}$.	1. Given.
2. $\frac{RT}{AM} = \frac{\frac{1}{2}RT}{\frac{1}{2}AM} = \frac{RS}{AQ}$.	2. Given $\overline{WS}$ and $\overline{LQ}$ are medians.
3. $\frac{RW}{AL} = \frac{WS}{LQ} = \frac{RS}{AQ}$.	3. Steps 1 and 2, and substitution.
4. $\Delta RSW \sim \Delta AQL$.	4. S.S.S. Similarity.
5. $\angle R \cong \angle A$.	5. Definition of similar triangles.
6. $\Delta RWT \sim \Delta ALM$.	6. Step 1 and Theorem 12-4.

19.

1. $\angle y$ is the complement of $\angle x$.	1. $\overline{RA} \perp \overline{AB}$, and definition of complementary angles.
2. $\angle y$ is the complement of $\angle R$.	2. Given $\overline{RH} \perp \overline{AF}$, and Corollary 9-13-2.
3. $\angle x \cong \angle R$.	3. Complements of the same angle are congruent.
4. $\angle B \cong \angle RHA$.	4. $\overline{RH} \perp \overline{AF}$ and $\overline{FB} \perp \overline{AB}$.
5. $\Delta HRA \sim \Delta BAF$.	5. A.A. Corollary.
6. $\frac{HR}{BA} = \frac{HA}{BF}$.	6. Definition of similar triangles.
7. $HR \cdot BF = BA \cdot HA$.	7. Clearing of fractions in Step 6.

386 20. a. No.

b. Bisect $\overline{PA_1}$, $\overline{PB_1}$, etc., and connect the resulting mid-points.

c. $\frac{PA_2}{PA_1} = \frac{PB_2}{PB_1}$ because both equal 2. $\angle A_1PB_1$ is common to triangles A_1PB_1 and A_2PB_2. These triangles are therefore similar by the S.A.S. Similarity Theorem; and as a result of their being similar the sides A_2B_2 and A_1B_1 have the same ratio as the other corresponding sides.

386 d. Not only A_2B_2 and A_1B_1, but other corresponding sides of triangles $A_2B_2D_2$ and $A_1B_1D_1$ are in the ratio 2:1 by a proof like that in part c. $\triangle A_2B_2D_2 \sim \triangle A_1B_1D_1$ by the S.S.S. Similarity Theorem.

e. Yes, the method could be used for any point P; but in some instances the enlargement would intersect the given figure.

387 *21. $\angle SRX \cong \angle QTX$ and $\angle RSX \cong \angle TQX$ (alternate interior $\angle$s), so $\triangle SRX \sim \triangle QTX$ by A.A. Therefore $\frac{RX}{TX} = \frac{SX}{QX}$, so $\frac{RX}{SX} = \frac{TX}{QX}$. Since $\triangle QXR \sim \triangle TXS$ (given), $\frac{RX}{SX} = \frac{QX}{TX}$. Therefore $\frac{QX}{TX} = \frac{TX}{QX}$, $(QX)^2 = (TX)^2$, and $QX = TX$, since both QX and TX are positive. $\angle XQR \cong \angle XTS$ and $\angle RXQ \cong \angle SXT$ (definition of similar triangles), so $\triangle QXR \cong \triangle TXS$ by A.S.A. Therefore $QR = TS$.

Alternate proof: If $TS > QR$, then $TX > QX$ and $XS > XR$, from $\triangle QXR \sim \triangle TXS$. In $\triangle QXT$, $m\angle XQT > m\angle XTQ$, by Theorem 7-4, and in $\triangle RXS$, $m\angle SRX > m\angle RSX$. But $m\angle XTQ = m\angle SRX$, by alternate interior $\angle$s, and $m\angle XQT = m\angle RSX$. Contradiction. Similarly if $QR > TS$.

387 22.

1.	$\overline{AW} \perp \overline{MW}$. BFRQ is a square.	1.	Given.
2.	$\angle ABQ \cong \angle W \cong \angle MFR$.	2.	Definitions of perpendicular and square.
3.	Let $m\angle A = a$ and $m\angle M = m$.	3.	Angle Measurement Postulate.
4.	Thus, $m\angle FRM = a$ and $m\angle AQB = m$.	4.	Corollary 9-13-2.
5.	Also, $m\angle WQR = a$ and $m\angle WRQ = m$.	5.	The sum of the measures of the angles at Q is 180 and the sum of the measures of the angles at R is 180.
6.	$\triangle ABQ \sim \triangle RFM \sim \triangle QWR$.	6.	A.A.A.
7.	$\frac{AB}{QW} = \frac{BQ}{WR}$ and $\frac{AB}{RF} = \frac{BQ}{FM}$.	7.	Definition of similar triangles.
8.	$AB \cdot WR = QW \cdot BQ$ and $AB \cdot FM = RF \cdot BQ$.	8.	Clearing of fractions in Step 7.

23. Since $\triangle ABF \sim \triangle HRQ$ we know $\angle F \cong \angle Q$ and $\frac{AF}{HQ} = \frac{AB}{HR} = \frac{BF}{RQ}$. Also $\frac{FB}{QR} = \frac{\frac{1}{2}FB}{\frac{1}{2}QR} = \frac{FW}{QX} = \frac{AF}{QH}$. Then $\triangle AWF \sim \triangle HXQ$ by S.A.S. Similarity, and then $\frac{AW}{HX} = \frac{AF}{HQ} = \frac{FB}{QR} = \frac{AB}{HR}$.

It is possible to continue in the same way for the other medians.

24. Since $\triangle ABF \sim \triangle XWR$ then $\angle x \cong \angle A$ and $\frac{XR}{AF} = \frac{XW}{AB} = \frac{WR}{BF}$. $m\angle AHF = m\angle XQR$ and so $\triangle XQR \sim \triangle AHF$ by A.A. Then $\frac{RQ}{FH} = \frac{XR}{AF}$.

A similar proof can be followed for each of the altitudes.

388 25. As shown in the two figures the two triangles are similar by A.A.

26. Since the base angles are congruent, $AE = BE$ and by subtraction $EC = ED$. Hence $\triangle CED \sim \triangle AEB$ by the S.A.S. Similarity Theorem.
Therefore $\angle ECD \cong \angle EAB$ and $\overline{CD} \parallel \overline{AB}$ by corresponding angles.

389 27. False. Let $\triangle AB_1C$ and $\triangle AB_2C$ be such that $AC = AC$, $\angle A \cong \angle A$, $CB_1 = CB_2$, as in the diagram, but the triangles are not congruent. Construct $\triangle A'B'C' \sim \triangle AB_1C$. The triangles $A'B'C'$ and AB_2C satisfy the statements of the hypothesis, but these triangles are not similar.

*28. a. 1. $\triangle ABC \sim \triangle ADE$; $\frac{AB}{AD} = \frac{AC}{AE} = \frac{BC}{DE}$.

2. $\triangle ABC$ and $\triangle ADF$ are not similar even though $\frac{AB}{AD} = \frac{BC}{DF}$ since $m\angle B \neq m\angle FDA$.

b. False. The diagram shows a counter-example. The hypothesis is true if X is either E or F. The conclusion is false if X is F.

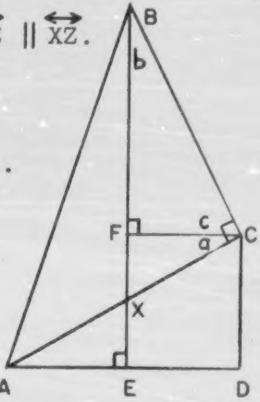
390 *29. In similar $\triangle$s ABC and EDC, $\frac{x}{39} = \frac{a}{b}$. From the similar $\triangle$s ACG and AEF,

$$\frac{a+b}{a} = \frac{7}{3},$$
$$1 + \frac{b}{a} = \frac{7}{3},$$
$$\frac{b}{a} = \frac{4}{3},$$
$$\frac{a}{b} = \frac{3}{4},$$
$$\frac{x}{39} = \frac{3}{4},$$
$$x = \frac{3}{4} \cdot 39 = 29\frac{1}{4}$$

Answer. The ball hits the ground at least 29'3" from the net.

390 *30. $\Delta CEB \sim \Delta AEF$ since $\angle x \cong \angle y$ (alternate interior angles of parallel lines $\overleftrightarrow{BC}$ and $\overleftrightarrow{AD}$) and $\angle FEA \cong \angle BEC$ (vertical angles); therefore $\frac{EF}{EB} = \frac{FA}{BC} = \frac{AF}{CE}$. Also, $\Delta CEG \sim \Delta AEB$ since $\angle ABE \cong \angle CGE$ (alternate interior angles) and $\angle CEG \cong \angle AEB$ (vertical angles); we get $\frac{BA}{GC} = \frac{AE}{CE} = \frac{EB}{EG}$. Since in each case we have $\frac{AE}{CE}$ as one of the fractions, we also have $\frac{EF}{EB} = \frac{EB}{EG}$.

*31. Since $\overleftrightarrow{AX} \parallel \overleftrightarrow{BY}$, $\Delta DAX \sim \Delta DBY$ and $\frac{DA}{DB} = \frac{AX}{BY}$. Similarly, since $\overleftrightarrow{CZ} \parallel \overleftrightarrow{BY}$, $\Delta CEZ \sim \Delta BEY$ and $\frac{EC}{EB} = \frac{CZ}{BY}$. But $AX = CZ$, since opposite sides of a parallelogram are congruent, and so $\frac{DA}{DB} = \frac{EC}{EB}$. Now $1 - \frac{DA}{DB} = 1 - \frac{EC}{EB}$, $\frac{DB - DA}{DB} = \frac{EB - EC}{EB}$ and $\frac{AB}{DB} = \frac{BC}{EB}$. Therefore $\overleftrightarrow{AC} \parallel \overleftrightarrow{DE}$ by Theorem 12-2. And now $\overleftrightarrow{AC} \parallel \overleftrightarrow{DE} \parallel \overleftrightarrow{XZ}$.



391 *32. a. In right Δs AXE and CXF, $\angle FXC \cong \angle EXA$, hence $\angle XAE \cong \angle XCF$ ($\angle a$). $\angle a$ is a complement of $\angle c$. $\angle b$ is a complement of $\angle c$. Hence $\angle a \cong \angle b \cong \angle XAE$. Hence $\Delta BFC \sim \Delta ADC$ and $\frac{BF}{BC} = \frac{AD}{AC}$.

b. Since AB occurs in each denominator, one only needs to show that

$$BE = CD + \frac{AD}{AC} \cdot BC\ .$$

Since

$$\begin{aligned} BE &= FE + BF \\ &= CD + BF \end{aligned}$$

one only needs to show that

$$BF = \frac{AD}{AC} \cdot BC\ .$$

This is essentially what was shown in part a. of this problem.

391 In Theorem 12-6 we have assumed the following theorem: In any right triangle the altitude from the vertex of the right angle intersects the hypotenuse in a point between the end-points of the hypotenuse.

Proof: Let D be the foot of the perpendicular from C to $\overleftrightarrow{AB}$.

391 There are 5 possible cases:

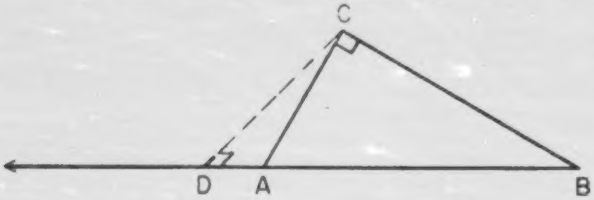
(1) D = A.
(2) D = B.
(3) A is between D and B.
(4) B is between D and A.
(5) D is between A and B.

We would like to show that cases (1), (2), (3), and (4) are impossible which leaves case (5) as the required result.

Case (1) is impossible because Δ BDC then would have two right angles, one at C and one at D.

Case (2) is impossible for a similar reason as in Case (1).

Proof that case (3) is impossible:

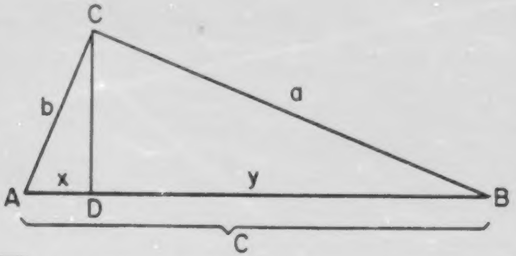


Suppose that A is between D and B. Then ∠ CDA is a right angle of Δ CDA. Moreover ∠ CAB is an exterior angle of Δ CDA and so is obtuse. But this is impossible, since ∠ CAB is an acute angle of Δ ABC.

A similar proof shows that Case (4) is impossible, hence, Case (5) holds as was to be proved and the altitude from C must intersect the hypotenuse at some point D, such that D is between A and B.

392 Once we have proved Theorem 12-6, it is now possible to prove the Pythagorean Theorem using similar triangles. This has not been done in the text, however, since the theorem has been proved once by areas. If time permits, it might be illuminating to the class to let them see the following proof, reminding them that there is more than one way to attack a mathematical problem.

Theorem: Given a right triangle, with legs of length a and b and hypotenuse of length c. Then $a^2 + b^2 = c^2$.



Proof: Let $\overline{CD}$ be the altitude from C to $\overline{AB}$, as in Theorem 12-6. Let $x = AD$ and let $y = DB$, as in the figure. The scheme of the proof is simple. (1) First we calculate x in terms of b and c, using similar triangles. (2) Then we calculate y in terms of a and c, using similar triangles. (3) Then we add x and y, and simplify the resulting equation, using the fact that $c = x + y$.

(1) Since $\Delta\ ACD \sim \Delta\ ABC$, we have $\frac{x}{b} = \frac{b}{c}$.
Therefore $x = \frac{b^2}{c}$.

(2) Since $\Delta\ CBD \sim \Delta\ ABC$, we have $\frac{y}{a} = \frac{a}{c}$.
Therefore $y = \frac{a^2}{c}$.

(3) Thus we have $x + y = \frac{a^2 + b^2}{c}$,
But $c = x + y$.
Therefore $c = \frac{a^2 + b^2}{c}$,
and $a^2 + b^2 = c^2$, which was to be proved.

395 Note to the teacher: At this point in the text you may wish to proceed directly to Chapter 17, Plane Coordinate Geometry, and later return to the remaining chapters.

Problem Set 12-4

393 1. $x = 2\sqrt{5}.$

$z = 6.$

$y = 3\sqrt{5}.$

2. $x = 16.$

$y = 4\sqrt{5}.$

$z = 8\sqrt{5}.$

394 3. a. $\frac{4}{6} = \frac{6}{x + 4}.$

$36 = 4x + 16.$

$20 = 4x.$

$5 = x.$

b. $\frac{4}{y} = \frac{y}{5}.$

$y^2 = 4 \cdot 5.$

$y = 2\sqrt{5}.$

c. $\frac{9}{a} = \frac{a}{5}.$

$a^2 = 9 \cdot 5.$

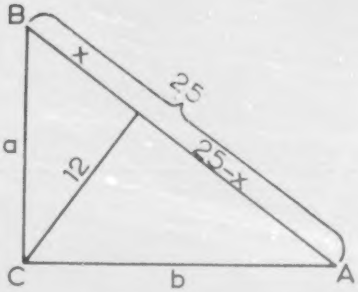
$a = 3\sqrt{5}.$

4. Let the segments of the hypotenuse be x and $25 - x$. Then $\frac{x}{12} = \frac{12}{25 - x}$ by Theorem 12-6 and definition of similar triangles.

$144 = 25x - x^2.$

$x^2 - 25x + 144 = 0.$

$(x - 9)(x - 16) = 0.$ The segments of the hypotenuse are 9 and 16. If a is the length of the shorter leg,



$\frac{25}{a} = \frac{a}{9}$,

$a = 15.$

$\frac{25}{b} = \frac{b}{16}$,

$b = 20.$

394 5. a. $CD = 4;\quad AC = \sqrt{20} = 2\sqrt{5};\quad CB = \sqrt{80} = 4\sqrt{5}.$

b. $DB = 27;\ AC = \sqrt{90} = 3\sqrt{10};\ CB = \sqrt{810} = 9\sqrt{10}.$

c. Let $DB = x$, then $x(x + 10) = 144.$

$$x^2 + 10x = 144.$$
$$x^2 + 10x - 144 = 0.$$
$$(x + 18)(x - 8) = 0.$$
$$x = 8.$$

$DB = 8.$

$CA = \sqrt{180} = 6\sqrt{5}.$

$CD = \sqrt{80} = 4\sqrt{5}.$

d. Let $AD = x$, then $x(x + 12) = 64.$

$$x^2 + 12x - 64 = 0.$$
$$(x - 4)(x + 16) = 0.$$
$$x = 4.$$

$AD = 4.$

$CB = 8\sqrt{3}.$

$CD = \sqrt{48} = 4\sqrt{3}.$

Problem Set 12-5

396 1. $\frac{9}{16}$; $\frac{x^2}{y^2}$.

2. $\frac{6}{25}$.

3. 4; $\frac{15}{4}$.

4. 3.

5. $(\frac{b}{20})^2 = \frac{36}{225} = (\frac{6}{15})^2 = (\frac{2}{5})^2,$

$\frac{b}{20} = \frac{2}{5}, \quad b = \frac{40}{5} = 8.$

The base of the smaller is 8 inches.

6. $\frac{9}{2}$.

7. Since $\overline{DE} \parallel \overline{AB}$, $\Delta ABC \sim \Delta DEC$.

$\frac{CA}{CD} = 3$ and so $\frac{\text{Area } \Delta \ ABC}{\text{Area } \Delta \ DEC} = 9.$

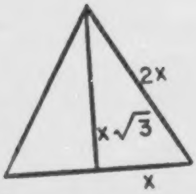
397 8. a. 2. b. 4.

9. $(\frac{S}{10})^2 = \frac{2}{1}$.

$\frac{S^2}{100} = \frac{2}{1}$.

$S^2 = 100 \cdot 2$.

$S = 10\sqrt{2}$. The sides will be $10\sqrt{2}$.



10. $\frac{A_1}{A_2} = (\frac{2x}{x\sqrt{3}})^2 = \frac{4}{3}$.

11. If the length of the wire is called d, the side of the square is $\frac{1}{4}d$ and that of the triangles is $\frac{1}{3}d$. Then the area of the square is $\frac{d^2}{16}$ and that of the triangle is $\frac{d^2}{36}\sqrt{3}$. Then,

$$\frac{\text{Area of the triangle}}{\text{Area of the square}} = \frac{\frac{d^2}{36}\sqrt{3}}{\frac{d^2}{16}} = \frac{4\sqrt{3}}{9}.$$

397 12. The area of Δ ABC $= \frac{1}{2}\cdot 140\cdot 120 = 8400$.

The area of the required lot must then be 4200. By the Pythagorean Theorem, AD = 90, and area of

Δ ADC $= \frac{1}{2}\cdot 90\cdot 120 = 5400$. Then, by Theorem 12-7,

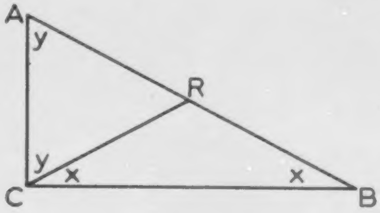
$(\frac{x}{90})^2 = \frac{4200}{5400}$, and $x = 30\sqrt{7}$. The required distance is approximately 79.4 feet.

13. Given: Right Δ ABC, $\angle$ C a right angle, and M the mid-point of $\overline{AB}$.

Prove: MA = MB = MC.

Proof: Let $\overline{MK}$ be the perpendicular from M to $\overline{BC}$, meeting $\overline{BC}$ in K. Then $\overline{MK} \parallel \overline{AC}$, so CK = KB. Therefore $\overleftrightarrow{MK}$ is the perpendicular bisector of $\overline{CB}$. Hence MC = MB. Since MB = MA (given), then MA = MB = MC.

398 14. By Problem 13, $KC = \frac{c}{2}$, where AB = c. Therefore $m\angle KCB = m\angle KBC = 60$, so $m\angle BKC = 60$. Therefore $BC = KB = \frac{c}{2}$.



15. Since AR = RC, $m\angle A = m\angle ACR$. Also, since RC = RB, $m\angle B = m\angle BCR$. Let $m\angle A = m\angle ACR = y$ and $m\angle B = m\angle BCR = x$. Then in Δ ACB, $2x + 2y = 180$, and $x + y = 90$.

*16. $HC = \sqrt{AH\cdot HB}$, $(HC)^2 = AH\cdot HB$, $\frac{AH}{HC} = \frac{CH}{HB}$.

Also $\angle AHC \cong \angle CHB$. Hence $\Delta AHC \sim \Delta CHB$ by S.A.S. Similarity Theorem. Therefore $\angle HCB \cong \angle A$. Since $\angle HCB$ and $\angle B$ are complementary, then $\angle A$ and $\angle B$ are complementary, and Δ ACB is a right triangle. By the preceding problem MC = AM, and $MC = \frac{1}{2}AB = \frac{1}{2}(AH + HB)$. But $HC < MC$, except when M = H (i.e., when AH = HB). Therefore, $\sqrt{AH\cdot HB} = HC < \frac{1}{2}(AH + HB)$. If AH = HB, the last inequality becomes the equality $\sqrt{(AH)^2} = \frac{1}{2}(AH + AH)$, that is $AH = AH$.

Alternate solution. Let u and v be positive numbers, $u \neq v$. Then

$$0 < (\sqrt{u} - \sqrt{v})^2 = u - 2\sqrt{u}\sqrt{v} + v.$$

$$2\sqrt{uv} < u + v.$$

$$\sqrt{uv} < \frac{u + v}{2}.$$

398 17. Outline of proof. $\Delta PXR \sim \Delta PYA$, therefore $\frac{PR}{PA} = \frac{PX}{PY}$.

$\Delta PRS \sim \Delta PAB$, therefore $\frac{PR}{PA} = \frac{RS}{AB}$.

$\Delta RST \sim \Delta ABC$, therefore

$$\frac{\text{Area } \Delta\ RST}{\text{Area } \Delta\ ABC} = \left(\frac{RS}{AB}\right)^2.$$

From the above:

$$\frac{\text{Area } \Delta\ RST}{\text{Area } \Delta\ ABC} = \left(\frac{PX}{PY}\right)^2.$$

399 *18.
1. Area Addition Postulate (Postulate 19).
2. Division.
3. Theorem 12-6.
4. Theorem 12-7 and Step 2.
5. Multiplication.

400 *19. $a^2 = h^2 + y^2 = h^2 + (c - x)^2.$

$$= (h^2 + x^2) + c^2 - 2cx.$$

$$= b^2 + c^2 - 2cx.$$

In the similarity $\Delta ADC \sim \Delta RST$,

$$\frac{x}{b} = \frac{k}{1} = k,$$

$$x = bk.$$

Therefore

$$a^2 = b^2 + c^2 - 2bck.$$

400 *20. $a^2 = h^2 + y^2 = h^2 + (x + c)^2.$

$$= (h^2 + x^2) + c^2 + 2cx.$$

$$= b^2 + c^2 + 2cx.$$

In the similarity $\Delta\ ADC \sim \Delta\ RST$,

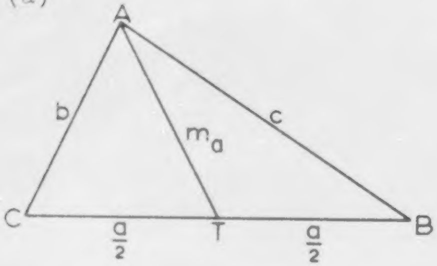
$$\frac{x}{b} = \frac{k}{1} = k,$$

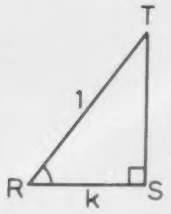
$$x = bk.$$

Therefore

$$a^2 = b^2 + c^2 + 2bck.$$

*21. (a)





(This is the case in which $\angle C$ is acute. If $\angle C$ is obtuse or a right angle, the proof is similar.)

Let Δ RST have $\angle R \cong \angle C$, $\angle S$ a right angle, hypotenuse = 1, RS = k. By the result of Problem 19, applied to Δ ACT,

$$m_a^2 = b^2 + (\tfrac{a}{2})^2 - 2b(\tfrac{a}{2})k,$$

(1) $$m_a^2 = b^2 + \frac{a^2}{4} - abk.$$

Applying the same result to Δ ACB,

(2) $$c^2 = b^2 + a^2 - 2abk.$$

Multiplying both sides of Equation (2) by $\frac{1}{2}$ and subtracting from the corresponding sides of Equation (1):

$$m_a^2 - \frac{1}{2}c^2 = \frac{b^2}{2} - \frac{a^2}{4},$$

$$m_a^2 = \frac{1}{2}(b^2 + c^2 - \frac{a^2}{2}).$$

400 (b) From part (a) $m_a{}^2 = \frac{1}{2}b^2 + \frac{1}{2}c^2 - \frac{1}{4}a^2$,

$$m_b{}^2 = \frac{1}{2}a^2 + \frac{1}{2}c^2 - \frac{1}{4}b^2,$$

$$m_c{}^2 = \frac{1}{2}a^2 + \frac{1}{2}b^2 - \frac{1}{4}c^2.$$

Adding and collecting like terms,

$$m_a{}^2 + m_b{}^2 + m_c{}^2 = \frac{3}{4}(a^2 + b^2 + c^2).$$

Review Problems

401 1. a. $\frac{2}{11} = \frac{4}{FB}$, hence $FB = 22$.

b. $\frac{1}{5} = \frac{FQ}{6}$, hence $FQ = \frac{6}{5}$.

c. $\frac{2\frac{1}{2}}{9} = \frac{FQ}{7}$, hence $FQ = \frac{35}{18}$.

d. $\frac{6}{9} = \frac{QB}{12}$, hence $QB = 8$.

2. a. Yes. b. $AF = 8$.

3. a. G.M. is $4\sqrt{5}$, A.M. is 9.

b. G.M. is 6, A.M. is $\frac{9}{2}\sqrt{2}$.

4. Sketches might be of two rhombuses; a rhombus and a square; two parallelograms; a parallelogram and a rectangle.

5. $\frac{12}{FC} = \frac{FC}{3}$, hence $FC = 6$. $\frac{12}{AC} = \frac{AC}{15}$, hence $AC = 6\sqrt{5}$.

$\frac{15}{BC} = \frac{BC}{3}$, hence $BC = 3\sqrt{5}$.

402 6. If $\overline{DE} \parallel \overline{AB}$, $\frac{x + 3}{3x + 3} = \frac{5}{x + 5}$, and $x = 7$.

402 7. $\Delta ABE \sim \Delta CDE$ (A.A.). Corresponding sides are therefore proportional and $DE = 4BE$. Hence $BD = 5BE$.

8. Let e be the length of the side of the original triangle. Then the length of the side of the second triangle is $\frac{e}{2}\sqrt{3}$ and the ratio of the areas is $\frac{4}{3}$.

9. $\frac{x}{8} = \frac{8}{20 - x}$; $x^2 - 20x + 64 = 0$; $x = 16$ or $x = 4$.

(i) If $x = 16$: $a^2 = 16^2 + 8^2$; $a = 8\sqrt{5}$;
$y = 20 - x = 4$; $b = 4\sqrt{5}$.

(ii) If $x = 4$: $a^2 = 4^2 + 8^2$; $a = 4\sqrt{5}$;
$y = 16$; $b = 8\sqrt{5}$.

Hence there are two possibilities: $x = 16$, $y = 4$, $a = 8\sqrt{5}$, $b = 4\sqrt{5}$ and $x = 4$, $y = 16$, $a = 4\sqrt{5}$, $b = 8\sqrt{5}$.

10. $\Delta ABC \sim \Delta DEF$, hence $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$.

$\Delta ACB \sim \Delta DEF$, hence $\frac{AC}{DE} = \frac{AB}{DF} = \frac{CB}{EF}$.

Since, above, the last ratios are the same, $\frac{AB}{DE} = \frac{AC}{DE}$ and hence $AB = AC$.

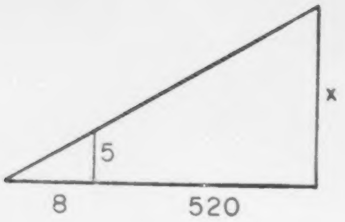
11. a. $\Delta AFQ \sim \Delta WAX$ (A.A.). Hence $\frac{AF}{WA} = \frac{AQ}{WX}$ and therefore $AF \cdot XW = AW \cdot QA$.

b. $\Delta AXW \sim \Delta FQA$ (A.A.) and so $\frac{QF}{AX} = \frac{QA}{XW}$; hence $QF \cdot XW = AX \cdot QA$.

c. Since $\Delta AXW \sim \Delta FQA$, $\frac{AW}{FA} = \frac{AX}{FQ}$, hence $AW \cdot FQ = FA \cdot AX$.

403 12. $\frac{5}{8} = \frac{x}{528}$.

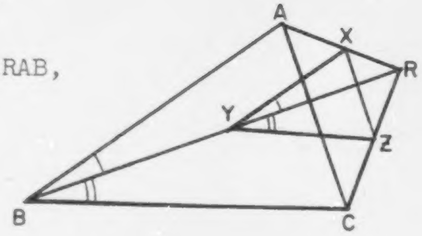
$x = 330$.



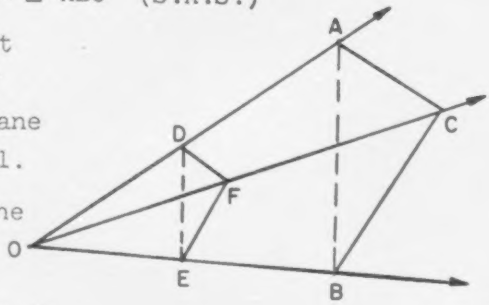
13. $\frac{3}{9} = \frac{9}{3 + y}$, hence $y = 24$. $\frac{3}{x} = \frac{x}{24}$, hence $x = 6\sqrt{2}$.

$\frac{27}{w} = \frac{w}{24}$, hence $w = 18\sqrt{2}$.

*14. $m\angle XYR = m\angle ABR$, $m\angle RYZ = m\angle RBC$ (corresponding angles.) By addition, $m\angle XYZ = m\angle ABC$. Since $\overline{XY} \parallel \overline{AB}$, $\triangle RXY \sim \triangle RAB$, hence $\frac{XY}{AB} = \frac{RY}{RB}$. Since $\overline{YZ} \parallel \overline{BC}$, $\triangle RYZ \sim \triangle RBC$, hence $\frac{RY}{RB} = \frac{YZ}{BC}$. Hence $\frac{XY}{AB} = \frac{YZ}{BC}$. Hence $\triangle XYZ \sim \triangle ABC$ (S.A.S.)



15. No. We can be sure that it is when the plane of the triangle and the plane of the film are parallel.



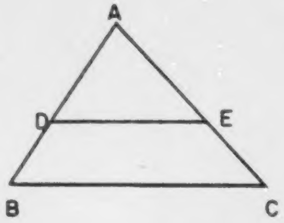
Proof: Assuming that the planes of $\triangle ABC$ and $\triangle DEF$ are parallel,
$\overline{DE} \parallel \overline{AB}$, $\overline{EF} \parallel \overline{BC}$, $\overline{DF} \parallel \overline{AC}$.
$\therefore \triangle ODE \sim \triangle OAB$,
$\triangle OEF \sim \triangle OBC$, $\triangle OFD \sim \triangle OCA$.
$\frac{EF}{BC} = \frac{OE}{OB} = \frac{ED}{BA} = \frac{OD}{OA} = \frac{DF}{AC}$, that is, $\frac{EF}{BC} = \frac{ED}{BA} = \frac{DF}{AC}$.

Therefore $\triangle ABC \sim \triangle DEF$ by S.S.S. Similarity.

Illustrative Test Items for Chapter 12

A. 1. a. In $\triangle ABC$, if $AD = 5$, $AB = 7$, $AE = 7\frac{1}{2}$, $EC = 3$, is $\overline{DE} \parallel \overline{BC}$? Explain.

b. In $\triangle ABC$, if $AD = 15$, $AB = 25$, $AC = 33$, and $AE = 21$, is $\overline{DE} \parallel \overline{BC}$? Explain.



2. a. Given two similar triangles in which the ratio of a pair of corresponding sides is $\frac{2}{3}$, what is the ratio of the areas?

b. If the ratio of the areas of two similar triangles is $\frac{1}{2}$, what is the ratio of a pair of corresponding altitudes?

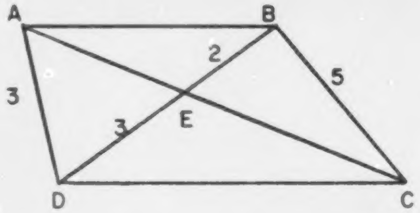
3. If 2, 5, 6 are the lengths of the sides of one triangle and $7\frac{1}{2}$, 9, 3 are the lengths of the sides of another triangle, are the triangles similar? If so, write ratios to show the correspondence of the sides.

4. If ABCD is a trapezoid with $\overline{AB} \parallel \overline{DC}$ and lengths of segments as shown, give numerical answers below:

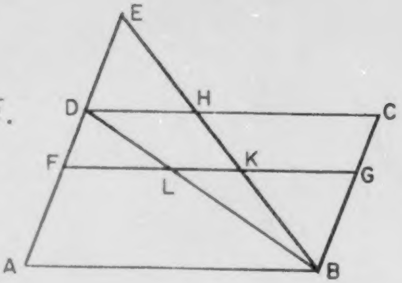
a. $\frac{AB}{CD} = ?$

b. $\frac{\text{Area } \triangle\ AEB}{\text{Area } \triangle\ CED} = ?$

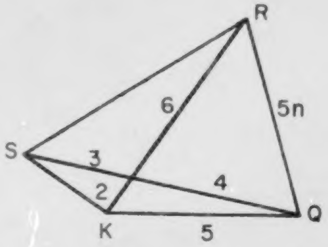
c. $\frac{\text{Area } \triangle\ ACD}{\text{Area } \triangle\ BDC} = ?$



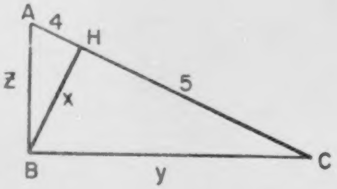
5. In the figure, ABCD is a parallelogram with $\overline{FG} \parallel \overline{DC}$. DF = 4, DE = 6, AB = 12, KB = 2·KH. Find AF, BC, DH, KF and LF.



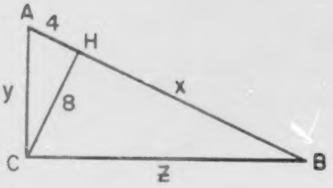
6. In quadrilateral KQRS in the figure, segments have lengths as shown. Find $\frac{KS}{SR}$ in terms of n.



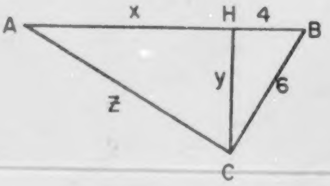
B. 1. In the figure, $\overline{AB} \perp \overline{BC}$, $\overline{BH} \perp \overline{AC}$, and the lengths of the segments are as shown. Find x, y, and z.



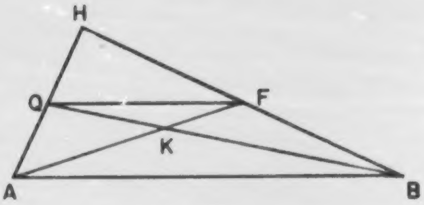
2. With $\overline{AC} \perp \overline{CB}$ and $\overline{CH} \perp \overline{AB}$ and with lengths as indicated in the figure, find x, y, and z.



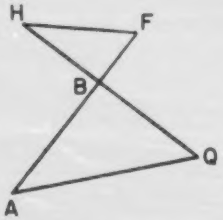
3. In this figure Δ ACB is a right triangle with altitude $\overline{HC}$ drawn to the hypotenuse $\overline{AB}$. Find x, y, and z.



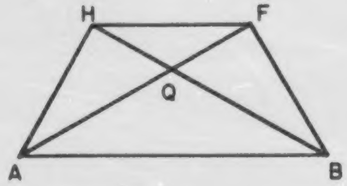
C. 1. $\overline{AF}$ and $\overline{BQ}$ are medians of Δ ABH, as shown in the figure. Prove $\Delta ABK \sim \Delta FQK$. Write three equal ratios showing the proportionality of the sides of these triangles, and give the numerical value of the ratios.



2. In this figure, $BF = \frac{2}{3}HB$ and $BQ = \frac{2}{3}AB$. Prove the two triangles are similar and write three equal ratios showing the proportionality of the sides.



3. $\overline{HF} \parallel \overline{AB}$ as shown in the figure. Prove $AB \cdot FQ = AQ \cdot FH$.



Answers

A. 1. a. Yes, since $\frac{5}{7} = \frac{7\frac{1}{2}}{10\frac{1}{2}}$ (Theorem 12-2).

b. No, since $\frac{15}{25} \neq \frac{21}{33}$.

2. a. $\frac{4}{9}$. b. $\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$.

3. Yes. $\frac{2}{3} = \frac{5}{7\frac{1}{2}} = \frac{6}{9}$.

4. a. $\frac{2}{3}$. b. $\frac{4}{9}$. c. 1.

5. AF = 8. BC = 12. DH = 4. KF = $\frac{20}{3}$. LF = 4.

6. KS = $\frac{5n}{2}$, RS = $\frac{15}{2}$, $\frac{KS}{SR} = \frac{n}{3}$.

B. 1. $\frac{4}{x} = \frac{x}{5}$, hence $x = 2\sqrt{5}$. $\frac{4}{z} = \frac{z}{9}$, hence $z = 6$.

$\frac{9}{y} = \frac{y}{5}$, hence $y = 3\sqrt{5}$.

2. $x = 16$. $y = 4\sqrt{5}$. $z = 8\sqrt{5}$.

3. $\frac{4}{6} = \frac{6}{x + 4}$, hence $x = 5$. $\frac{5}{y} = \frac{y}{4}$, hence $y = 2\sqrt{5}$.

$\frac{5}{z} = \frac{z}{9}$, hence $z = 3\sqrt{5}$.

C. 1. $\angle AKB \cong \angle FKQ$ (vertical angles) and
$\angle BQF \cong \angle QBA$ (alternate interior angles),
hence $\triangle AKB \sim \triangle FKQ$ (A.A.) $\frac{FK}{AK} = \frac{FQ}{AB} = \frac{KQ}{KB} = \frac{1}{2}$.

2. Since $\frac{BF}{HB} = \frac{2}{3} = \frac{BQ}{AB}$ and $\angle HBF \cong \angle ABQ$, $\triangle HBF \sim \triangle ABQ$ (S.A.S.) and $\frac{HB}{AB} = \frac{BF}{BQ} = \frac{HF}{AQ}$.

3. $\triangle ABQ \sim \triangle FHQ$ (A.A.) and $\frac{AB}{FH} = \frac{AQ}{FQ}$, hence $AB \cdot FQ = AQ \cdot FH$.

Answers to Review Exercises
Chapters 7 to 12

1. 0.
2. 1.
3. 1.
4. 0.
5. 1.
6. 0.
7. 0.
8. 0.
9. 0.
10. 1.
11. 1.
12. 1.
13. 1.
14. 0.
15. 0.
16. 1.
17. 0.
18. 0.
19. 1.
20. 0.
21. 1.
22. 1.
23. 0.
24. 1.
25. 1.
26. 0.
27. 1.
28. 1.
29. 0.
30. 0.
31. 0.
32. 0.
33. 1.
34. 1.
35. 1.
36. 0.
37. 1.
38. 0.
39. 1.
40. 0.
41. 1.
42. 0.
43. 1.
44. 0.
45. 0.
46. 1.
47. 1.
48. 0.
49. 0.
50. 0.
51. 1.
52. 1.
53. 1.
54. 1.
55. 0.
56. 1.
57. 1.
58. 0.
59. 0.
60. 0.

[pages 404-408]

Chapter 13

CIRCLES AND SPHERES

This chapter falls into two parts: the first studies common properties of circles and spheres relative to intersection with lines and planes, the second deals with degree measure of circular arcs and related properties of angles and arcs, chords, secants and tangents. The first part is unusual since it treats circles and spheres by uniform methods and states and proves the fundamental theorems on the intersection of line and circle (and sphere and plane) with great precision. You will note that following the fundamental theorems on circles, there is a corresponding section concerning spheres, and probably nowhere else is the analogy between plane and space geometry so strong as it is here. Essentially the same proofs work for the sphere as the circle, as relates to tangent and secant lines and planes. The theorems and methods of proof in the second part are, in the main, conventional but the basic ideas of types of circular arc, angles inscribed in an arc, and arc intercepted by an angle are defined with unusual care.

411 The convention of letting circle P mean the circle with center P is followed in many of the problems for convenience, where no ambiguity results. The text, however, follows the more precise notation, where a separate letter denotes the circle. We can then talk concisely about concentric circles C and C' or about line L intersecting circle C.

Use concrete situations to illustrate the idea of circle and sphere. For example, ask students to describe the figure composed of all points which are six inches from a given point of the blackboard - but don't say "points of the blackboard". Use models, cut a ball in half to indicate its center and radius, and so on. Refer to the earth and the equator (or meridians) as examples of a sphere and a great circle. Contrast "great circle" with "small circle", such as the equator with a parallel north of the equator.

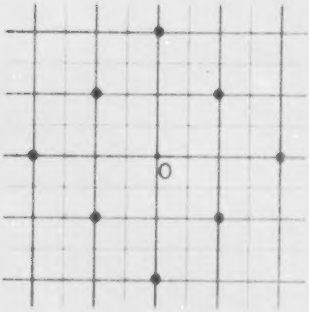
Problem Set 13-1

411 1. a. False.
b. True.
c. False.
d. False.
e. False.
f. True.
g. True.
h. True.

2. a. False.
b. True.
c. True.
d. False.
e. False.
f. False.
g. True.
h. False.

412 3. a. All points lie on a circle with center at the given intersection, and radius 200 yards.

b. There are eight such points: four of them lie at the vertices of a square, and four at the midpoints of the sides of this square, as shown on the diagram. (O is the given intersection.)



4. Let c be the length of any chord not a diameter. Draw radii to its end-points. Then $2r > c$, by Theorem 7-7, The Triangle Inequality. But $2r$ is the length of the diameter. Hence the diameter is larger than any other chord.

---

412 We have not adopted the convention that the distance from a point to itself shall be zero - that is, the distance between points is always a positive number. For this reason, in defining the interior of a circle (or sphere), we must include the center in addition to points whose distance to the center is less than the radius.

414 Cases (1) and (2) of Theorem 13-2 should be easy for students to grasp. In Case (2), the answer to "Why?" is Theorem 7-6 (The perpendicular segment is the shortest distance from a point to a line).

415 Case (3), (see below) is more difficult and may cause trouble for some students - also they may think it hair splitting to prove something so "obvious". If they learn and understand the theorem and omit the proof of Case (3), they still may be better off than in a conventional course in which the precise relation between lines and circles is not made explicit, let alone proved. Incidentally, Theorem 13-5 is an exact analog of Theorem 13-2, but is less familiar and less obvious. After working through the proof of Theorem 13-5 they may better appreciate the proof of Theorem 13-2.

415 Remark on Theorem 13-2, Case (3): Case (3) is essentially the same as an existence and uniqueness proof. Since we don't know that L and C have points in common, we assume they have a common point and try to find where it can possibly lie. Precisely we try to locate it relative to F which is a fixed point on L.

Thus in the first part of the proof we show:

If a point is common to L and C its distance from F is $\sqrt{r^2 - PF^2}$. Since $\sqrt{r^2 - PF^2}$ is a definite positive number, we see that there are only two possible positions on L for a point common to L and C, namely the two points on L whose distance to F is $\sqrt{r^2 - PF^2}$.

415 In the second part we show a converse: If a point is on L and its distance from F is $\sqrt{r^2 - PF^2}$ then it is common to L and C. To show this we merely show that $PQ = r$, as follows:

$$PQ = \sqrt{FQ^2 + PF^2} = \sqrt{r^2 - PF^2 + PF^2} = \sqrt{r^2} = r.$$

Thus the two points described above are common to L and C and constitute their intersection.

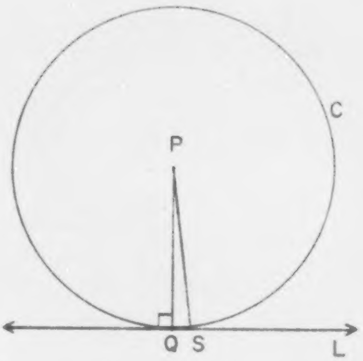
415 If your students prefer to derive some of these corollaries by using congruent triangles and other earlier principles rather than Theorem 13-2, by all means permit them to do so. The fact that Theorem 13-2 is a powerful theorem may be seen better in retrospect by many students.

In applying Theorem 13-2 (and Theorem 13-5) we generally show that since two of the cases do not hold in a particular situation the other one must hold.

Proofs of the Corollaries

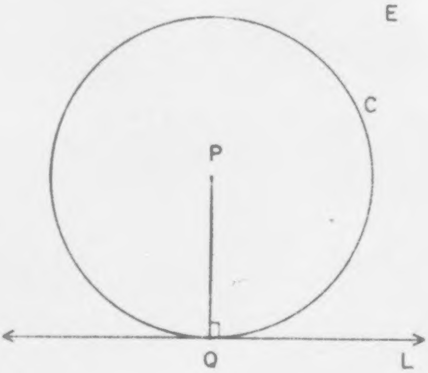
Corollary 13-2-1. Any line tangent to C is perpendicular to the radius drawn to the point of contact.

Let L be a tangent to C at point S. Draw the radius $\overline{PS}$. Let Q be the foot of the perpendicular from P to L. If $Q \neq S$, then L intersects C in exactly 2 points and this contradicts the hypothesis that L is tangent to C at S. Therefore the point Q must be the point S, hence the tangent L is perpendicular to the radius drawn to the point of contact.

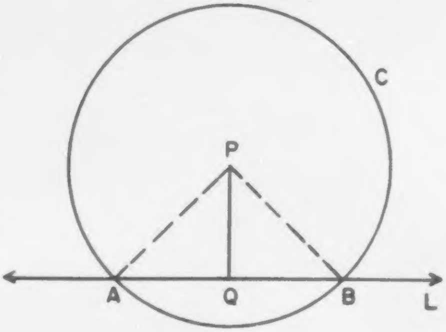


416 Corollary 13-2-2. Any line in E perpendicular to a radius at its outer end, is tangent to the circle.

Given a line in E, perpendicular to a radius at its outer end, which is a point on circle C. This point is Q, the foot of the perpendicular from center P to L. Then, by Theorem 13-2, the line intersects the circle in Q alone and is therefore tangent to the circle.

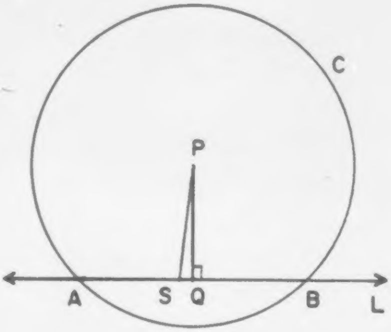


Corollary 13-2-3. Any perpendicular from the center of C to a chord bisects the chord.



Consider a chord $\overline{AB}$ of circle C and the line L containing $\overline{AB}$. The line L intersects C in two points A and B. Let Q be the foot of the perpendicular from P to L. The intersection cannot be Q alone. Hence, by Theorem 13-2, A and B are equidistant from Q. Therefore the perpendicular from P to the chord bisects the chord.

Corollary 13-2-4. The segment joining the center of a circle to the mid-point of a chord is perpendicular to the chord.



416 Given chord $\overline{AB}$ of circle C and segment $\overline{PS}$ where P is the center of circle C and S is the mid-point of chord $\overline{AB}$. Let $\overline{PQ} \perp \overline{AB}$ with foot Q. By Corollary 13-2-3, Q is the mid-point of $\overline{AB}$. Since the mid-point of $\overline{AB}$ is unique $(Q = S)$, $\overline{PS}$ is perpendicular to the chord $\overline{AB}$.

Alternate Proof: Let F be the mid-point of $\overline{AB}$. Then P and F are equidistant from A and B in plane E and $\overleftrightarrow{PF}$ is the perpendicular bisector of $\overline{AB}$ in plane E by Theorem 6-2.

This also can be done independently of Theorem 13-2 by using congruent triangles.

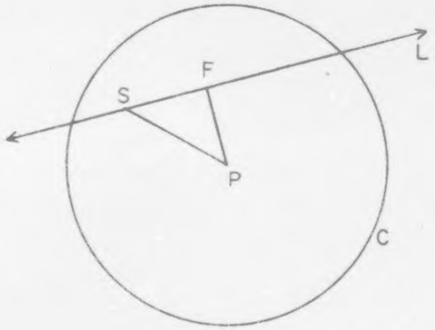
Corollary 13-2-5. In the plane of a circle, the perpendicular bisector of a chord passes through the center of the circle.

By Corollary 13-2-4 the segment joining the center of a circle to the mid-point of a chord is perpendicular to the chord, hence the line containing the center of a circle and the mid-point of the chord is *a* perpendicular bisector of the chord. Since there is only *one* perpendicular to the chord at its mid-point, the perpendicular bisector of a chord must pass through the center of the circle.

416 Alternate Proof: The perpendicular bisector of the chord in the plane of the circle contains all points of this plane which are equidistant from the end-points of the chord (Theorem 6-2). Therefore the perpendicular bisector contains the center.

Corollary 13-2-6. If a line in the plane of a circle intersects the interior of the circle, then it intersects the circle in exactly two points.

Consider line L in the plane E of circle C which contains a point S inside C. Let F be the foot of the perpendicular from P to L. By Theorem 7-6, $PF \le PS$. Since S is in the interior of C, $PS < r$. Hence, $PF < r$, and so F is in the interior of C and Condition (3) holds.



Note on Corollary 13-2-6. This corollary differs from Case (3) of Theorem 13-2 in that the point in the interior of C does not have to be F, the foot of the perpendicular to the line. Probably most students will consider this difference quite unimportant, and a proof of an obvious fact as very superfluous. While you may not care to bring it up, a significance of this corollary is that it indicates the precision of our treatment of circles using Theorem 13-2 which allows us to give a formal proof of such an intuitively obvious result.

417 The idea of congruent circles gives you an excellent opportunity to discuss the general idea of congruence. Point out that to say two figures are congruent means that they can be made to "fit" or that one is an exact copy of the other. But it is very difficult to give the student a precise mathematical definition of the idea until he knows a fair amount of geometry (see Appendix on Rigid Motion). Therefore we define congruence piecemeal for segments, angles, triangles, circles, arcs of circles and so on. But in each case we frame the definition to ensure that the figures are congruent, that is, "can be made to fit". So in the present case, we define circles to be congruent if they have congruent radii not because we consider this condition to be the basic idea, but because we are intuitively certain that it guarantees that the circle can be made to fit.

417 It might be well to remind the students of what is involved in the concept of the distance between a point and a line, including the case where the distance is zero.

Note that in the proof of Theorem 13-3 we have assumed that the distance from each chord to the center is not zero. If it is zero, each chord is a diameter and the theorem still holds.

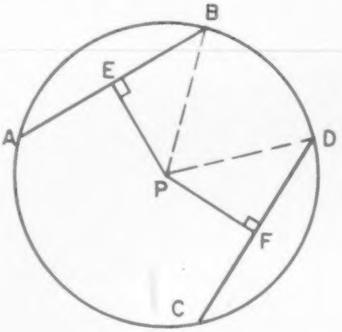
Proofs of Theorems 13-3 and 13-4

Theorem 13-3. In the same circle or congruent circles, chords equidistant from the center are congruent.

Given: Chords $\overline{AB}$ and $\overline{CD}$, equidistant from P.

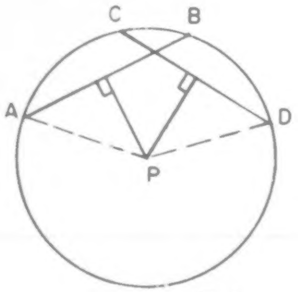
To prove: $\overline{AB} \cong \overline{CD}$.

Let $\overline{PE} \perp \overline{AB}$ and $\overline{PF} \perp \overline{CD}$ as in the figure. Draw radii $\overline{PB}$ and $\overline{PD}$. Then in right triangles PEB and PFD we have:



(1) PE = PF.	(1) Given.
(2) $\overline{PB} \cong \overline{PD}$.	(2) Radii of same or congruent circles are congruent.
(3) Δ PEB $\cong \Delta$ PFD.	(3) Hypotenuse and Leg Theorem.
(4) EB = FD.	(4) Corresponding parts.
(5) EB = $\frac{1}{2}$AB. FD = $\frac{1}{2}$CD.	(5) Corollary 13-2-3.
(6) $\frac{1}{2}$AB = $\frac{1}{2}$CD.	(6) Substitution.
(7) AB = CD or $\overline{AB} \cong \overline{CD}$.	(7) Algebra.

Note that this proof still holds if $\overline{AB}$ intersects $\overline{CD}$ as shown below:

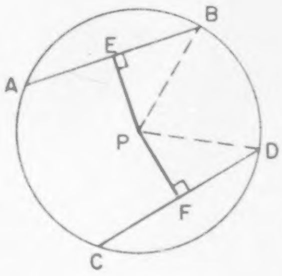


Proof of Theorem 13-4: In the same circle or congruent circles, any two congruent chords are equidistant from the center.

Given: Chords $\overline{AB} \cong \overline{CD}$. P is the center of the circle.

To prove: PE = PF where $\overline{PE} \perp \overline{AB}$ and $\overline{PF} \perp \overline{CD}$ as in the figure.

Draw radii $\overline{PB}$ and $\overline{PD}$.



(1) $\overline{PB} \cong \overline{PD}$.	(1) Radii of same or congruent circles are congruent.
(2) AB = CD.	(2) Given.
(3) $\frac{1}{2}AB = \frac{1}{2}CD$.	(3) Multiplication, Step 2.
(4) $EB = \frac{1}{2}AB$. $FD = \frac{1}{2}CD$.	(4) Corollary 13-2-3.
(5) EB = FD.	(5) Steps 3 and 4.
(6) Δ PEB $\cong$ Δ PFD.	(6) Hypotenuse-Leg Theorem.
(7) $\overline{PE} \cong \overline{PF}$ or PE = PF.	(7) Corresponding parts.

417 As in the conventional treatment we have implicitly assumed that the distances of the chords from center P are not zero. If both distances are zero, the chords are diameters and the theorem is correct. Could one distance be zero and the other not? The answer of course is no, and is justified by the following minor theorem: A diameter is the longest chord of a circle. (See Problem Set 13-1, Problem 4.)

In this chapter there are very many interesting results of the theorems in the text proper. Many of these interesting facts are to be found in the problem sets, accompanied

[page 417]

by problems providing numerical application of the fact. In assigning problems, teachers should be careful to watch for such sequences and select accordingly.

Problem Set 13-2

418 1.
- a. Corollary 13-2-4.
- b. Corollary 13-2-2.
- c. Corollary 13-2-6.
- d. Corollary 13-2-5.
- e. Theorem 13-3.
- f. Corollary 13-2-1.
- g. Corollary 13-2-3.
- h. Theorem 13-4.

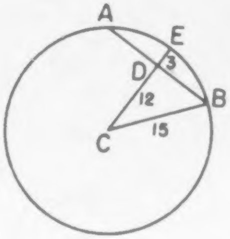
2. (See Teacher's Commentary for proof of Corollary 13-2-3.)

3. (See Teacher's Commentary for proof of Corollary 13-2-5.)

4. By Corollary 13-2-5, the perpendicular bisector of a chord passes through the center of the circle. Hence, to find the center draw any two chords in the circle and the perpendicular bisector of each. The intersection of these bisectors will be the center of the circle.

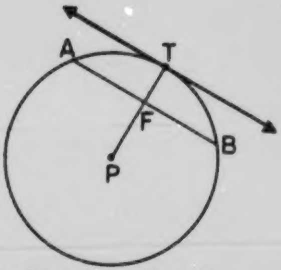
419 5. Draw a perpendicular from C to $\overline{MN}$, forming a 3-4-5 right triangle. Then the distance from C to $\overline{MN}$ is 16.

6. As in the figure, CB = 15 and DC = 12. Then DB = 9, and the chord is 18 inches long.



419 7.
a. D.
b. C.
c. C.
d. A.
e. C.
f. A.
g. B.
h. D.
i. C.
j. D.

420 8. Let $\overline{PT}$ intersect $\overline{AB}$ at F. Then FB = 6. Δ BFP is a 30 - 60 right triangle. Hence PB = $4\sqrt{3}$.



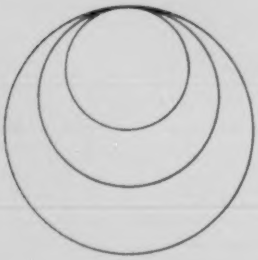
420 9. Since a tangent to a circle is perpendicular to the radius drawn to the point of contact, the two tangents will be perpendicular to the same line and are, therefore, parallel.

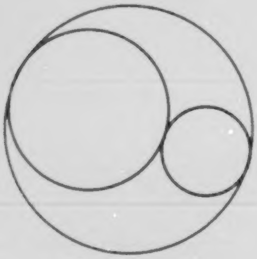
*10.

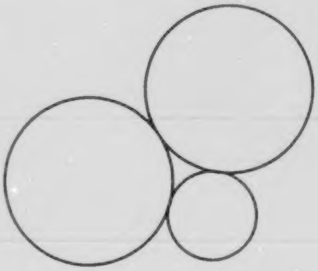
1.	$\overleftrightarrow{DO} \parallel \overleftrightarrow{AC}$. $\overleftrightarrow{CD}$ is tangent at C.	1.	Given.
2.	$\angle A \cong \angle BOD$.	2.	Corresponding angles of parallels.
3.	OC = OA = OB.	3.	Definition of circle.
4.	$\angle A \cong \angle ACO$.	4.	Theorem 5-2.
5.	$\angle ACO \cong \angle COD$.	5.	Alternate interior angles.
6.	$\angle COD \cong \angle BOD$.	6.	Steps 2, 4, and 5.
7.	OD = OD.	7.	Identity.
8.	Δ OCD $\cong$ Δ OBD.	8.	S.A.S. and Steps 3, 6, and 7.
9.	$\angle OCD \cong \angle OBD$.	9.	Definition of congruent triangles.
10.	$m\angle OCD = 90$.	10.	Corollary 13-2-1.
11.	$m\angle OBD = 90$.	11.	Steps 9 and 10.
12.	$\overleftrightarrow{DB}$ is tangent at B.	12.	Corollary 13-2-2.

420 11. Draw $\overline{OR}$. $\overline{OR} \perp \overline{AB}$, by Corollary 13-2-1. AR = BR, by Corollary 13-2-3.

421 12. Here are three arrangements.





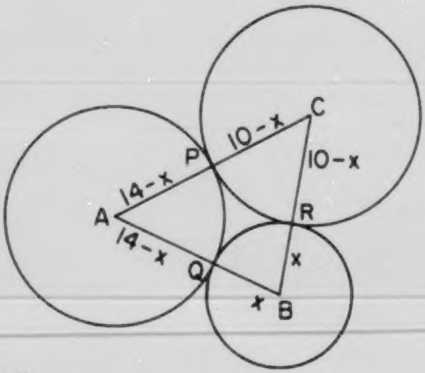


*13. Let L be the common tangent. Then in both cases, $\overline{PT} \perp L$ and $\overline{QT} \perp L$ by Corollary 13-2-1. But there exists only one perpendicular to a line at a point on the line. Hence $\overline{PT}$ and $\overline{QT}$ are collinear. This means that P, Q, and T are collinear.

14. $AC = 14 - x + 10 - x = 18.$

$$24 - 2x = 18.$$
$$2x = 6.$$
$$x = 3.$$

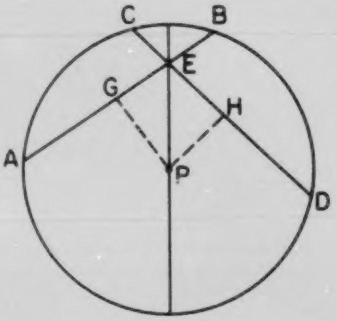
BR = 3, CP = 7,
AQ = 11.



422 *15. (See Teacher's Commentary for proof of Theorem 13-3.)

16. Given: $\angle AEP \cong \angle DEP$.

Prove: $\overline{AB} \cong \overline{CD}$.



Draw $\overline{PG} \perp \overline{AB}$ and $\overline{PH} \perp \overline{CD}$. Then $\triangle PGE$ and $\triangle PHE$ are right triangles with $m\angle GEP = m\angle HEP$, and $EP = EP$. Hence, $\triangle PGE \cong \triangle PHE$, making $PG = PH$. By Theorem 13-3, $\overline{AB} \cong \overline{CD}$.

17. Since $RD = RE$, $AB = BC$ by Theorem 13-3. But $DA = \frac{1}{2}AB$ and $EC = \frac{1}{2}BC$ by Corollary 13-2-3. Hence, $DA = EC$.

18. By Corollary 13-2-4 the segment joining a mid-point of a chord to the center is perpendicular to the chord. By Theorem 13-3 these segments all have equal lengths. By the definition of a circle, all points equidistant from a point lie on the circle having the point as center and its radius equal to the distance. By Corollary 13-2-2 the chords are all tangent to the inner circle.

*19.

1. $\overline{AO} \cong \overline{OB}$.	1. Definition of a circle.
2. $\overleftrightarrow{OT} \perp \overleftrightarrow{CD}$.	2. Corollary 13-2-1.
3. $\overleftrightarrow{AC} \perp \overleftrightarrow{CD}$, $\overleftrightarrow{BD} \perp \overleftrightarrow{CD}$.	3. Given.
4. $\overleftrightarrow{AC} \parallel \overleftrightarrow{OT} \parallel \overleftrightarrow{BD}$.	4. Theorem 9-2.
5. $\overline{CT} \cong \overline{TD}$.	5. Theorem 9-26.
6. $m\angle CTO = m\angle DTO = 90$.	6. Perpendicular lines form right angles.
7. $\overline{OT} \cong \overline{OT}$.	7. Identity.
8. $\triangle CTO \cong \triangle DTO$.	8. S.A.S.
9. $\overline{CO} \cong \overline{DO}$.	9. Corresponding parts.

423 Notice now closely the basic theorem on secant and tangent planes, Theorem 13-5, follows the pattern of Theorem 13-2, the basic theorem on secant and tangent lines of a circle. As in the case of Theorem 13-2, the point Q plays a major role in Theorem 13-5 and its corollaries.

425 Note that to prove (3) we show that two sets are identical; that is, the intersection of E and S is the same set as the circle with center F and radius $\sqrt{r^2 - x^2}$. This is why there are two parts to prove: (1) If Q is in the intersection then Q is in the circle; and conversely, (2) if Q is in the circle then Q is in the intersection. (Compare the discussion of the alleged identity of the Yale Mathematics Department and the Olympic Hockey Team of the Commentary, Chapter 10.)

Observe that we establish (1) and (2) by showing:

(1') If a point is common to E and S its distance from F is $\sqrt{r^2 - PF^2}$.

(2') If a point is in E and its distance from F is $\sqrt{r^2 - PF^2}$ then it is common to E and S.

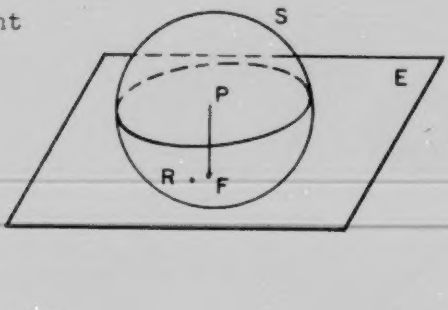
Compare with Case (3) of Theorem 13-2.

Proofs of the Corollaries

426 Corollary 13-5-1. Every plane tangent to S is perpendicular to the radius drawn to the point of contact.

Given: Plane E tangent to S at point R.

To prove: Plane E perpendicular to the radius drawn to the point of contact.

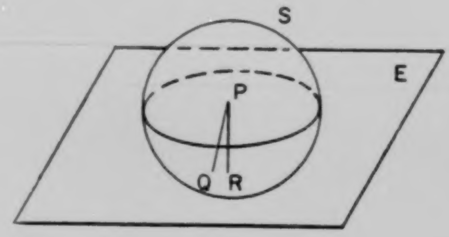


We will use the same method as in Corollary 13-2-1. Let F be the foot of the perpendicular from P to E. Since E is tangent to S and meets it in only one point, Cases (1) and (3) of Theorem 13-5 do not apply. Therefore (2) applies so that F is on S and E is tangent to S at F. Therefore $\overline{PF}$ is the radius drawn to the point of contact and $E \perp \overline{PF}$.

426 Corollary 13-5-2. Any plane perpendicular to a radius at its outer end is tangent to S.

Given: Plane E is perpendicular to radius $\overline{PR}$ at R.

To prove: Plane E is tangent to S. Then R is the foot of the perpendicular to plane E from P. By Theorem 13-5, plane E intersects S only at R, hence, E is tangent to S.



Corollaries 13-5-3 and 13-5-4 are actually not corollaries to Theorem 13-5 since their proofs do not require the theorem. They are easily proved and are placed here simply for convenience.

Corollary 13-5-3. A perpendicular from P to a chord of S, bisects the chord.

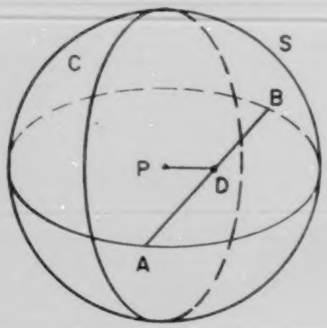
By Theorem 13-1, the plane determined by P and $\overline{AB}$ intersects S in a great circle. Then applying Corollary 13-2-3 we get AQ = BQ.

A proof using congruent triangles is also possible.

Corollary 13-5-4. The segment joining the center to the mid-point of a chord is perpendicular to the chord.

Given: Sphere S with
D the mid-point of chord $\overline{AB}$.
P is the center of S.

To prove: $\overline{PD} \perp \overline{AB}$.



As in Corollary 13-5-3, the plane PAB intersects S in a great circle. Then $\overline{PD} \perp \overline{AB}$ by Corollary 13-2-4.

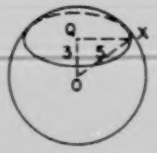
Other proofs are possible.

Problem Set 13-3

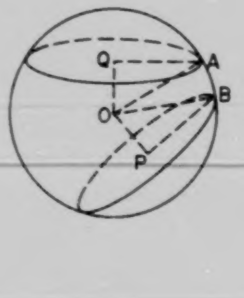
427 1. $\overleftrightarrow{OA} \perp \overleftrightarrow{PB}$.
$\overleftrightarrow{OA} \perp \overleftrightarrow{RT}$.

2. By Corollary 13-5-3, the perpendicular bisects the chord. By Pythagorean Theorem, one-half the chord is 8, so the length of the chord is 16.

3. By the Pythagorean Theorem, QX = 4 inches.



4. $\overline{OQ}$ and $\overline{OP}$ are perpendicular to the planes of the circles. Therefore $\overline{OQ} \perp \overline{QA}$ and $\overline{OP} \perp \overline{PB}$. OA = OB, by the definition of sphere, and OQ = OP, by hypothesis. Then, by the Pythagorean Theorem, QA = PB. Hence circle Q ≅ circle P, by definition.



427 *5. AF = BF since they are radii of the circle of intersection, and OF = AF by hypothesis. Also, $\overline{OF} \perp \overline{AF}$, $\overline{OF} \perp \overline{BF}$, and $\overline{AF} \perp \overline{BF}$. Hence, $\triangle AFB \cong \triangle AFO \cong \triangle BFO$, and $\triangle AOB$ is equilateral. Therefore AO = 5, $m\angle AOB = 60$, and OG, the altitude of $\triangle AOB$, equals $\frac{5}{2}\sqrt{3}$.

*6. Call the three points A, B, C. To find the center of the circle, in the plane ABC construct the perpendicular bisectors of any two of the three segments $\overline{AB}$, $\overline{BC}$, $\overline{AC}$. The bisectors intersect at the center, Q, of the circle. $\overline{QA}$, $\overline{QB}$, or $\overline{QC}$ is a radius of the circle. Construct the perpendicular to plane ABC at Q. This perpendicular meets the sphere in two points, X and Y. Determine the mid-point, P, of $\overline{XY}$. P is the center of the sphere. $\overline{PA}$, $\overline{PB}$, or $\overline{PC}$ is a radius of the sphere.

428 *7. By Theorem 13-5 we know that plane F intersects S in a circle. By Postulate 8, the two planes intersect in a line. Since both intersections contain T, the circle and line intersect at T. If they are not tangent at T, then they would intersect in some other point, R, also. Point R would then lie in plane E and in sphere S. But this is impossible, since E and S are tangent at T. Hence, the circle and the line are tangent, by definition.

8. By definition, a great circle lies in a plane through the center of the sphere. The intersection of the two planes must contain the center of the sphere, so that the segment of the intersection which is a chord of the sphere is a diameter of the sphere, and also of each circle.

428 *9. The plane of the perpendicular great circle is the plane perpendicular to the line of intersection of the planes of the given two, at the center of the sphere. There is only one such plane, by Theorem 8-9. Any two meridians have the equator as their common perpendicular.

*10. The intersection of the spheres is a circle. This can be shown as follows: Let M and M' be any points of the intersection. Then $\triangle AMB \cong \triangle AM'B$ by S.S.S. If $\overline{MO}$ and $\overline{M'O'}$ are altitudes from M and M', $\triangle AMO \cong \triangle AM'O'$ by A.A.S., so that $AO = AO'$ and $O = O'$. Hence all points M lie on a plane perpendicular to $\overleftrightarrow{AB}$ at O and on a circle with center O and radius OM. Since A and B are each equidistant from M and N, then all points on $\overleftrightarrow{AB}$ are equidistant from M and N, by Theorem 8-1, and $\overleftrightarrow{AB}$ is perpendicular to the plane of the intersection, by the argument above. By Theorem 11-10, we have $MO = 5$ in $\triangle MOB$. In $\triangle MOA$, by Pythagorean Theorem, we get $AO = 12$. But $OB = 5$. Hence $AB = 17$.

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430 Caution the students that they will be finding the degree measure of arcs and not the length of arcs.

432 If $\overset{\frown}{AC}$ is a minor arc then the theorem follows from The Angle Addition Postulate. (Postulate 13)

432 It may be noted that if $\overset{\frown}{AC}$ is a semi-circle, the theorem follows immediately from The Supplement Postulate (Postulate 14). The proof of the general case, though more troublesome, is made to depend upon these two cases. For a complete proof of Theorem 13-6 see Chapter 8 of Studies II.

432 In the definition of an angle inscribed in an arc, it is important to get across to the student that we are talking about angles inscribed in arcs of circles. Two points separate the circle into two arcs. The student should see that if an angle is inscribed in one of the arcs, the vertex is on that arc and the angle intercepts the other arc. In many geometry texts this is abbreviated to "an angle inscribed in a circle", but this can only mean "inscribed in an arc of a circle", since this is the way it has been defined in the text.

433 Condition (2) for an intercepted arc says, "each side of the angle contains an end-point of the arc". Notice that in the 4th example, in the preceding figures if one side is tangent to the circle, the other side of the angle contains both end-points of the intercepted arc and the tangent contains one end-point. For a discussion of Theorem 13-7 see Studies II.

435 The "Why?" in the first case is the Angle Addition Postulate; in the second case it is Theorem 13-6.

437-440 In Problem Set 13-4a, Problems 1 and 6 define two terms which you may want students to be familiar with. Also, Problems 5, 6, 10, 11 and 12 point up interesting facts.

Problem Set 13-4a

437 1. The center is the intersection of the perpendicular bisectors of two or more chords of the arc. (See Problem 4 of Problem Set 13-2.)

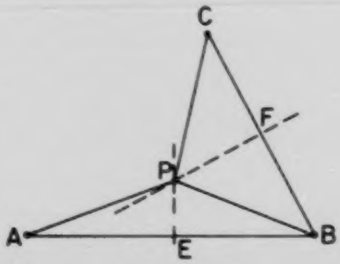
2. Since an inscribed angle is measured by half the arc it intercepts, $\widehat{AB}$ must contain 90°. Since the measure of a central angle is the measure of its intercepted arc, $m\angle P = 90$ and $\overline{BP} \perp \overline{AP}$.

437 3. a. $m\angle A = m\angle B$ by Corollary 13-7-2.
$m\angle AHK = m\angle BHF$ since the intercepted arcs have equal measure. Therefore $\triangle AHK \sim \triangle BHF$ by the A.A. Corollary.

b. $\triangle BFK$, since $m\angle BFA = \frac{1}{2}m\overset{\frown}{AB} = \frac{1}{2}m\overset{\frown}{BF} = m\angle BHF$, and $\angle HBF$ is common to the triangles.

438 4. Draw $\overline{RO}$. We know that $\overline{AO}$ is a diameter of the smaller circle and therefore that $m\angle ARO = 90$, by Corollary 13-7-1. Then $\overline{AB}$ is bisected by the smaller circle at point R, by Corollary 13-2-3.

*5. Draw $\overline{AB}$ and $\overline{BC}$ and draw the perpendicular bisector of each segment. Since the segments $\overline{AB}$ and $\overline{BC}$ are not parallel or collinear, the perpendicular bisectors are not parallel and therefore intersect in a point P.



This can be seen by using Theorem 9-12, Theorem 9-2, and the Parallel Postulate, in that order. $AP = BP$, and $BP = CP$ by Theorem 6-2. Hence $AP = BP = CP$. By definition of circle, A,B,C must lie on a circle with center P.

6. $m\angle C = \frac{1}{2}m\overset{\frown}{DAB}$.
$m\angle A = \frac{1}{2}m\overset{\frown}{DCB}$.

Since the sum of these two arcs is the entire circle, $m\angle C + m\angle A = 180$. Similarly, $m\angle B + m\angle D = 180$.

7. $m\overset{\frown}{ST} = 80$,
$m\overset{\frown}{RV} = 150$,
$m\angle T = 95$,
$m\angle V = 60$,
$m\angle S = 120$.

439 8. By Problem 6, $\angle C$ and $\angle BXY$ are supplementary and $\angle D$ and $\angle AXY$ are supplementary. But $\angle AXY$ and $\angle BXY$ are supplementary. Therefore $\angle D$ and $\angle C$ are supplementary and so $AD \parallel BC$.

9. Draw radii $\overline{PA}$ and $\overline{PB}$. Since $\overline{CD} \perp \overline{AB}$, $AM = BM$ by Corollary 13-2-3. $\triangle APM \cong \triangle BPM$ by S.S.S. (or S.A.S. or Hypotenuse-Leg), so that $m\angle APC = m\angle BPC$. Also, $m\angle APD = m\angle BPD$ by supplements of congruent angles. Therefore $m\overset{\frown}{AC} = m\overset{\frown}{BC}$ and $m\overset{\frown}{AD} = m\overset{\frown}{BD}$, by the definition of measure of an arc. Hence $\overline{CD}$ bisects $\overset{\frown}{ACB}$ and $\overset{\frown}{ADB}$.

10. $\triangle ACB$ is a right triangle with right angle at C, by Corollary 13-7-1. CD is the geometric mean of AD and BD, by Corollary 12-6-1.

11. By Theorem 13-7, $m\angle A = \frac{1}{2} m\overset{\frown}{BDC}$. Since $m\angle A = 90$, then $m\overset{\frown}{BDC} = 180$, and $\overset{\frown}{BDC}$ is a semi-circle. Hence, by definition, $\overset{\frown}{BAC}$ is a semi-circle.

440 *12. By Problem 5 we know there is a circle through A,B,C. Let $\overleftrightarrow{CD}$ intersect this circle in D'. Then ABCD' is inscribed in the circle, and, by Problem 6, $\angle BAD'$ is supplementary to $\angle C$. But $\angle BAD$ is supplementary to $\angle C$ by hypothesis. Therefore, $\angle BAD' \cong \angle BAD$, since supplements of the same angle are congruent. Hence, $\overleftrightarrow{AD} = \overleftrightarrow{AD'}$ and $D = D'$.

*13. Since $\overline{AC}$ and $\overline{BD}$ are tangent at the end-points of a diameter, then $\overline{AC} \parallel \overline{BD}$. Also, $\overline{AC}$ and $\overline{BD}$ are segments of chords in the larger circle which are congruent by Theorem 13-3. By Corollary 13-2-3, the radii $\overline{OA}$ and $\overline{OB}$ bisect these chords, so that $\overline{AC} \cong \overline{BD}$. Therefore quadrilateral ADBC is a parallelogram, by Theorem 9-20. But the diagonals of a parallelogram bisect each other, so that $\overline{AB}$ and $\overline{CD}$ bisect each other at some point, P. Now O is the mid-point of $\overline{AB}$, so $P = O$, and C, O, D are collinear, making $\overline{CD}$ a diameter.

Other proofs are possible.

---

441 Theorem 13-9. In the same circle or in congruent circles, if two arcs are congruent, then so also are the corresponding chords.

Using the figure in the text for Theorem 13-8 we see that:

Given: $\widehat{AB} \cong \widehat{A'B'}$.

To prove: $AB = A'B'$.

Since $\widehat{AB} = \widehat{A'B'}$, $\angle P \cong \angle P'$, and by S.A.S. Postulate we have $\triangle APB \cong \triangle A'B'P'$. Therefore $AB = A'B'$, by corresponding parts. If $\widehat{AB}$ and $\widehat{A'B'}$ are major arcs the same conclusion holds. If the arcs are semi-circles then the chords are diameters and are congruent.

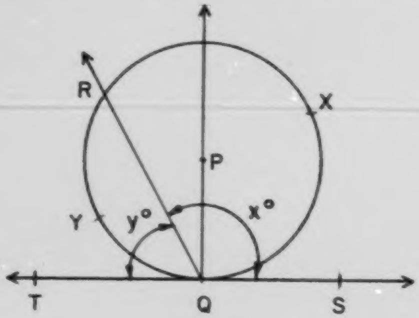
442 Theorem 13-10 is immediate if $\angle SQR$ is a right angle, since then the intercepted arc is a semi-circle.

Here is a proof for Theorem 13-10 in the case in which $\angle SQR$ is obtuse.

Given: $\angle SQR$ is obtuse.

To prove: $m\angle SQR = \frac{1}{2}m\widehat{QXR}$.

Let $\overrightarrow{QT}$ be the ray opposite to $\overrightarrow{QS}$. Let x and y be the measures of $\angle SQR$ and $\angle TQR$, as in the figure.



1.	$y = \frac{1}{2}m\widehat{QYR}$.	1.	Theorem 13-10, Case in text.
2.	$x = 180 - y$.	2.	Supplement Postulate.
3.	$x = 180 - \frac{1}{2}m\widehat{QYR}$.	3.	Steps 1 and 2.
4.	$x = \frac{1}{2}(360 - m\widehat{QYR})$.	4.	Algebra.
5.	$x = \frac{1}{2}m\widehat{QXR}$.	5.	Definition of measure of a major arc.

443 In Problem Set 13-4b, Problems 8, 9, 10, 14 and 16 are interesting theorems in their own right and are applicable to many numerical problems. They are easily grasped and proved. However, they are not essential to later deductive proof in the text.

In the theorems on these pages we will be establishing relationships about the products of the lengths of segments by first establishing a proportion involving these segments using similar triangles.

Problem Set 13-4b

443 1. (See Teacher's Commentary for proof of Theorem 13-9.)

2. a. Since chords $\overline{AF}$ and $\overline{BH}$ are congruent, they cut off congruent minor arcs $\widehat{HAB}$ and $\widehat{FBA}$. By Theorem 13-6, $m\widehat{HA} + m\widehat{AB} = m\widehat{FB} + m\widehat{AB}$, and so $m\widehat{HA} = m\widehat{FB}$.

   b. From $m\widehat{HA} = m\widehat{FB}$ we get $HA = FB$ by Theorem 13-9. $m\angle A = m\angle B$ and $m\angle AHB = m\angle BFA$ by Corollary 13-7-2. Then $\triangle AMH \cong \triangle BMF$ by A.S.A.

3. Since $ABCD$ is a square, $\overline{DA} \cong \overline{AB} \cong \overline{BC}$, and therefore, $\widehat{DA} \cong \widehat{AB} \cong \widehat{BC}$ by Theorem 13-8. Then $m\angle DEA = m\angle AEB = m\angle BEC$ since they are inscribed angles which intercept congruent arcs in the same circle.

4. a. $\angle BAC$.
   b. $\angle CAF$.
   c. $\angle ADB$, $\angle BAF$.
   d. $\angle DAF$.
   e. $\angle DCB$.
   f. $\angle ADC$.
   g. $\angle DCA$, $\angle DBA$.
   h. $\angle DAF$.
   i. $\angle EAB$.
   j. $\angle DBC$.

444 5. Since $m\widehat{PB} = 120$, $m\angle BPC = 60$ by Theorem 13-10. $\overline{PQ} \perp \overline{CP}$, so that $m\angle BPQ = 30$. $\triangle APQ$ is a 30 - 60 right triangle. Hence, $AP = 4\sqrt{3}$.

*6. Draw the common tangent at H. Then the angle formed by the tangent at H and line u is measured by the same arc as the angles formed by the line u and the tangents at M and N. Then the tangents at M and N are parallel by corresponding angles in one case and by alternate interior angles in the other case.

*7. Draw $\overline{PB}$. By Theorem 13-7, $m\angle BPR = \frac{1}{2}m\widehat{BR}$. By Theorem 13-10, $m\angle BPT = \frac{1}{2}m\widehat{PB}$. But $m\widehat{BR} = m\widehat{PB}$, so $m\angle BPR = m\angle BPT$. $\overline{BF} \perp \overleftrightarrow{PT}$ and $\overline{BE} \perp \overleftrightarrow{PR}$ by definition of distance from a point to a line. $PB = PB$, so $\triangle PBE \cong \triangle PBF$ by A.A.S. Therefore, $BE = BF$, which was to be proved.

445 8. Draw $\overline{BC}$, forming $\triangle BCE$. Then, $m\angle DEB = m\angle C + m\angle B$ $= \frac{1}{2}m\widehat{DB} + \frac{1}{2}m\widehat{AC} = \frac{1}{2}(m\widehat{DB} + m\widehat{AC})$.

9. Draw $\overline{BC}$, forming $\triangle BCE$. Then, $m\angle E = m\angle ABC - m\angle C$ $= \frac{1}{2}m\widehat{AC} - \frac{1}{2}m\widehat{BD} = \frac{1}{2}(m\widehat{AC} - m\widehat{BD})$.

10. The proof is the same as for Problem 9, except that Theorem 13-10 is used to get the measure of one angle in each case.

11. $m\widehat{BC} = 30$. $m\angle BAD = 30$.
$m\widehat{CD} = 30$. $m\angle AGE = 70$.
$m\angle K = 25$. $m\angle DGE = 110$.
$m\angle E = 30$. $m\angle ADK = 140$.

446 12. $m\widehat{DA} = 88$ and $m\widehat{BC} = 122$.
$m\angle EDC = m\angle DBC = 31$.
$m\angle CMD = m\angle AMB = m\angle ABC = 75$.
$m\angle DMA = m\angle CMB = 105$.
$m\angle FDB = m\angle DCB = 88$.
$m\angle ACB = m\angle ACD = m\angle DBA = 44$.
$m\angle CAB = m\angle CDB = 61$.
$m\angle DCE = m\angle BDE = 92$.
$m\angle DEC = 57$.
$m\angle DFA = 48$.
$m\angle CAF = 119$.
$m\angle CDF = 149$.
$m\angle ACE = 136$.

13. a. By Corollary 13-7-2, $m\angle ADP = m\angle BCP$ and $m\angle DAP = m\angle CBP$. Hence $\triangle APD \sim \triangle BPC$ by A.A.

b. Since similar triangles have corresponding sides proportional, $\frac{AP}{PB} = \frac{PD}{PC}$. Clearing of fractions we have $AP \cdot PC = PB \cdot PD$.

14. a. By Theorem 13-10, $m\angle DAC = \frac{1}{2}m\widehat{AC}$, and by Theorem 13-7, $m\angle B = \frac{1}{2}m\widehat{AC}$. Therefore $m\angle DAC = m\angle B$. Since $\angle D$ is common to the triangles, $\triangle ABD \sim \triangle CAD$ by A.A.

b. Since similar triangles have corresponding sides proportional, $\frac{BD}{AD} = \frac{AD}{CD}$. Clearing of fractions we have $BD \cdot CD = AD^2$.

447 *15. $m\angle a = \frac{1}{2}(m\overparen{AV} - m\overparen{DU}) = \frac{1}{2}(m\overparen{VB} - m\overparen{UC})$, so $m\overparen{AV} + m\overparen{UC} = m\overparen{VB} + m\overparen{DU}$. Similarly, working with $\angle b$, $m\overparen{SD} + m\overparen{BT} = m\overparen{AS} + m\overparen{TC}$.

Now
$$\begin{aligned} m\angle PRQ &= \tfrac{1}{2}(m\overparen{UT} + m\overparen{SV}) = \tfrac{1}{2}(m\overparen{UC} + m\overparen{CT} + m\overparen{AS} + m\overparen{AV}) \\ &= \tfrac{1}{2}(m\overparen{UC} + m\overparen{AV}) + \tfrac{1}{2}(m\overparen{CT} + m\overparen{AS}) \\ &= \tfrac{1}{2}(m\overparen{VB} + m\overparen{DU}) + \tfrac{1}{2}(m\overparen{SD} + m\overparen{BT}) \\ &= \tfrac{1}{2}(m\overparen{VB} + m\overparen{BT}) + \tfrac{1}{2}(m\overparen{SD} + m\overparen{DU}) \\ &= \tfrac{1}{2}(m\overparen{VT} + m\overparen{SU}) = m\angle QRV. \end{aligned}$$

Therefore $\angle PRQ$ is a right angle, by definition.

*16. Case I: Draw the diameter from P. Since the diameter is perpendicular to the tangent it is perpendicular to $\overleftrightarrow{AB}$, By Theorem 9-12. Therefore, $m\overparen{AP} = m\overparen{BP}$.

Case II: Draw the diameter perpendicular to the secants. By Case I, $m\overparen{AP} = m\overparen{BP}$ and $m\overparen{CP} = m\overparen{DP}$. By subtraction, $m\overparen{AC} = m\overparen{BD}$.

Case III: The diameter from P will have Q as its other end-point, by Theorem 9-12 and Theorem 13-2. Then the two arcs are semi-circles having equal measures, by definition.

Alternate proofs involve drawing radii to form congruent triangles, or drawing chords which are transversals and using alternate interior angles.

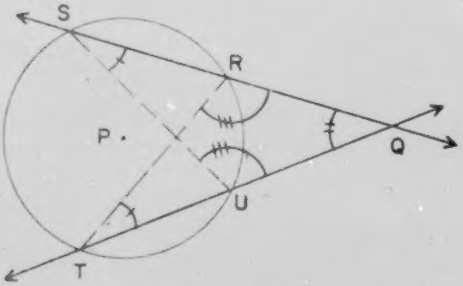
---

450 Theorem 13-13 is sometimes stated, "Given a tangent and a secant to a circle from an external point, the length of the tangent is the geometric mean of the length of the secant and the length of its external segment." The reasons in the proof are as follows: (1) Theorem 13-7; (2) Theorem 13-10; (3) Substitution; (4) Corollary 12-3-1 ($\angle Q = \angle Q$, Identity); (5) Corresponding sides of similar triangles are proportional; (6) Multiply both sides by $QR \cdot QT$.

451 Theorem 13-14 stated in words is, "If two chords intersect within a circle, the product of the lengths of the segments of one equals the product of the lengths of the segments of the other."

If the labeling of the figures for Theorems 13-12, 13-13, and 13-14 is kept consistent as illustrated below $\triangle SQU \sim \triangle TQR$ in each case.

$\triangle SQU \sim \triangle TQR$
Theorem 13-12
$QR \cdot QS = QU \cdot QT$

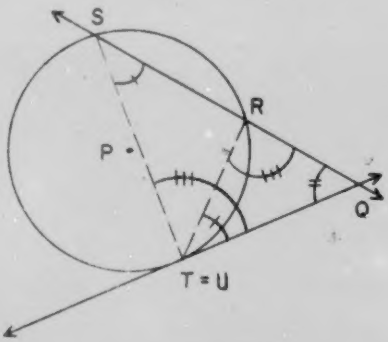


$\triangle SQU \sim \triangle TQR$
Notice that we used U as the name of the point of contact of the tangent $\overleftrightarrow{QT}$ in $\triangle SQU$ and used T as the name in $\triangle TQR$. Theorem 13-3, $QR \cdot QS = QT^2$

$QR \cdot QS = QU \cdot QT$

but since $T = U$

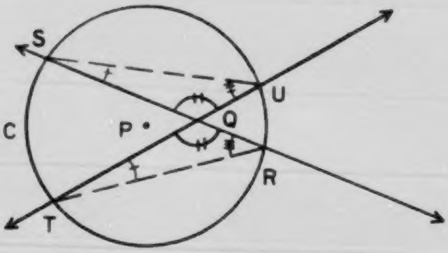
$QR \cdot QS = QU^2 = QT^2$.



$\Delta SQU \sim \Delta TQR$

Theorem 13-14

$QR \cdot QS = QU \cdot QT$



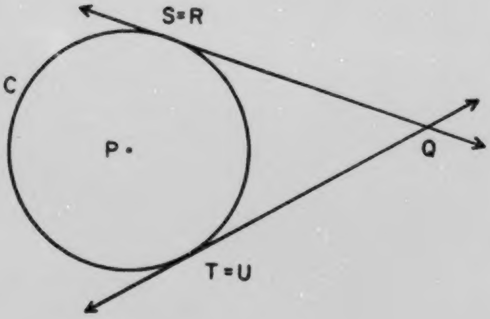
$QR \cdot QS = QU \cdot QT$

Since $R = S$ and $T = U$

we get $QR \cdot QR = QT \cdot QT$

$QR^2 = QT^2$.

Since QR and QT are positive numbers we have Theorem 13-11, $QR = QT$.



$QR \cdot QS = QU \cdot QT$.

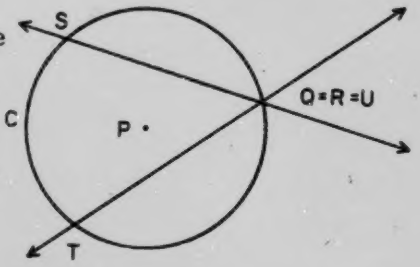
Since $Q = R = U$ then

$QR = 0$ and $QU = 0$, hence

$0 \cdot QS = 0 \cdot QT$

$0 = 0$

and this is a trivial result, but the pattern $QR \cdot QS = QU \cdot QT$ still holds.



Problem Set 13-5

452 1.

1.	$\overleftrightarrow{AC}$, $\overleftrightarrow{CE}$ and $\overleftrightarrow{EH}$ are tangents at B, D, and F respectively.	1.	Given.
2.	CB = CD. EF = ED.	2.	Theorem 13-11.
3.	CB + EF = CD + DE = CE.	3.	Addition.

2. By Theorem 13-12, $x(x + 13) = 4 \cdot 12$.

$$\begin{aligned} x^2 + 13x &= 48. \\ x^2 + 13x - 48 &= 0. \\ (x + 16)(x - 3) &= 0. \\ x &= 3. \end{aligned}$$

3. Let BK = a. Then by Theorem 13-13,

$$\begin{aligned} a(a + 5) &= 36. \\ a^2 + 5a - 36 &= 0. \\ (a + 9)(a - 4) &= 0. \\ a &= 4. \end{aligned}$$

$BK = 4.$

453 4. By Theorem 13-14, we have

$$\begin{aligned} x(19 - x) &= 6 \cdot 8. \\ x^2 - 19x + 48 &= 0. \\ (x - 3)(x - 16) &= 0. \\ x &= 3. \end{aligned}$$

$w = 19 - x = 16.$

453 5.

1. $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ are tangent at A and C, respectively.	1. Given.
2. Δ AOB and Δ COB are right triangles.	2. Corollary 13-2-1.
3. $m\angle ABO = m\angle CBO = 60$.	3. $m\angle ABC = 120$, and Theorem 13-11.
4. $AB = \frac{1}{2}OB$; $CB = \frac{1}{2}OB$.	4. Theorem 11-9.
5. $AB + CB = OB$.	5. Addition.

6. Since tangents to a circle from an external point are congruent,

$$SN = SP,$$
$$NR = RM,$$
$$CL = CP,$$
$$DL = DM.$$

Adding and grouping,

$(SN + NR) + (CL + DL) = (SP + CP) + (RM + DM)$, or

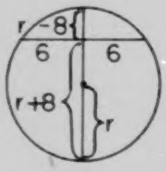
$SR + CD = SC + RD$.

7. Let r be the radius.

Then, by Theorem 13-14,

$(r + 8)(r - 8) = 6 \cdot 6$,

$r^2 - 64 = 36$, $r = 10$.



8. By Theorem 13-12, $7 \cdot AR = 6 \cdot 14$,

$AR = 6 \cdot 2 = 12$.

$AB = 12 - 7 = 5$.

454 9. Let the radius of the circle be r. Then by Theorem 13-13, $4(2r + 4) = (12)^2$. Hence $r = 16$.

454 10. Since all angles of the triangle have a measure of 60 the minor arc has a measure of 120. This leaves 240 for the measure of the major arc.

11. If m is the length of the shortest of the four segments, the rest of its chord would have to be the longest of the segments. Otherwise the product of the segments of this chord would certainly be less than the product of the segments of the other. Hence, if it were possible to have consecutive integers for the lengths they would be labeled as shown. But in this case, by Theorem 13-14, it would be necessary that:

$$m(m + 3) = (m + 1)(m + 2)$$

or $$m^2 + 3m = m^2 + 3m + 2$$

or $$0 = 2.$$

Since this is impossible, the lengths of the segments cannot be consecutive integers.

*12. Applying Theorem 13-13, we have $AM^2 = MR \cdot MS$ and $MB^2 = MR \cdot MS$. Hence, $AM^2 = MB^2$ and $AM = MB$. Similarly $CN = ND$.

455 13. a. Four; two internal, two external.

b. One internal, two external.

c. Two external only.

d. One external only.

e. None.

455 *14. Draw radii $\overline{RA}$ and $\overline{QB}$. Let $\overline{AB}$ intersect $\overline{RQ}$ at P. $m\angle A = m\angle B = 90$, and $m\angle APR = m\angle BPQ$ by vertical angles. Therefore $\Delta APR \sim \Delta BPQ$ by A.A. This gives $\frac{RP}{QP} = \frac{RA}{QB}$. Now suppose $\overline{DC}$ meets $\overline{RQ}$ at point P'. Then, by a similar argument, we arrive at $\frac{RP'}{QP'} = \frac{RA}{QB}$. Hence $\frac{RP'}{QP'} = \frac{RP}{QP}$, and P and P' are both between R and Q. Therefore $P' = P$.

*15. Problem 14 assures us that $\overline{AB}$ and $\overline{CD}$ meet $\overline{RQ}$ at the same point P. Therefore, by Theorem 13-11, $PA = PC$ and $PB = PD$. Adding, we have $PA + PB = PC + PD$, or $AB = CD$.

456 16. Draw $\overline{QR} \perp \overline{AP}$. In ΔPQR, $RQ = \sqrt{(PQ)^2 - (PR)^2}$. Hence $RQ = 48$. But $AB = RQ$, since RQBA is a rectangle. Therefore, $AB = 48$.

17. As in the previous problem, draw a perpendicular from the center of the smaller circle to a radius of the larger circle. By the Pythagorean Theorem, the distance between the centers is 39 inches.

18. Draw $\overline{QE} \perp \overleftrightarrow{PA}$. Since $PQ = 20$ and $PE = 7 + 9 = 16$, then $QE = 12 = AB$.

*19. Let d be the required distance. By Theorem 13-13

$$d^2 = \frac{h}{5280}\left(8000 + \frac{h}{5280}\right).$$

$$d^2 = \frac{50}{33}h + \left(\frac{h}{5280}\right)^2.$$

Now since h is very small compared to 5280, $\left(\frac{h}{5280}\right)^2$ is exceedingly small, and is not significant. So approximately, $d = \sqrt{1.515h} = 1.23\sqrt{h}$.

Hence, d is roughly $\frac{5}{4}\sqrt{h}$.

Review Problems

457 1.
- a. chord.
- b. diameter. (also chord.)
- c. secant.
- d. radius.
- e. tangent.
- f. minor arc.
- g. major arc.
- h. inscribed angle.
- i. central angle.

2. 55 and 70.

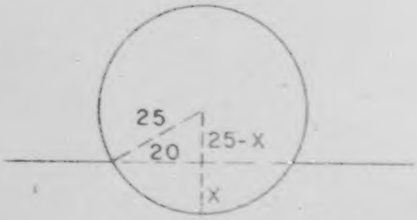
3. $m\angle AXB = 90$, because it is inscribed in a semi-circle. $m\angle AXY = 45$. $m\widehat{AY} = 90$ since $\angle AXY$ is inscribed. Hence the measure of central angle ACY is 90 making $\overline{CY} \perp \overline{AB}$.

458 4.
- a. True.
- b. True.
- c. False.
- d. True.
- e. False.
- f. True.
- g. False.
- h. True.
- i. True.
- j. True.

5. $m\angle C = 65$. $m\angle ABX = 65$.

459 6. Let $m\widehat{HE} = r$. Then $m\angle PCH = 90 - r$,
$m\angle NHC = 180 - (90 - r)$ or $90 + r$. Then
$m\angle NHR = m\angle NHC - 90 = (90 + r) - 90 = r$.
Hence, $m\widehat{HE} = m\angle NHR$.

7. The figure shows a cross-section with x the depth to be found.



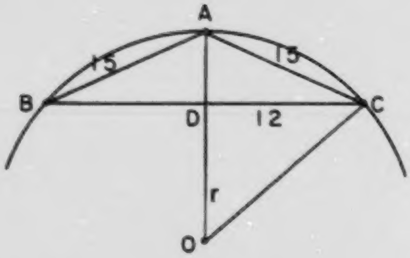
$25^2 = 20^2 + (25 - x)^2$

$225 = (25 - x)^2$

$15 = 25 - x$

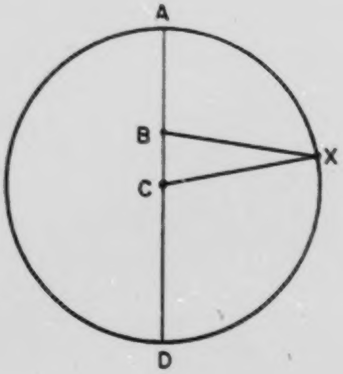
$10 = x$. The depth is 10 inches.

459 8. By the Pythagorean Theorem, $AD = 9$. If r is the radius, then $OD = r - 9$ and $OC = r$. Hence, in ΔDOC,

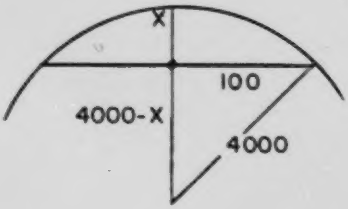


$$r^2 = (r - 9)^2 + 12^2,$$
$$r^2 = r^2 - 18r + 81 + 144,$$
$$18r = 225,$$
$r = 12.5$. The diameter of the wheel is 25 inches long.

9. Consider the distance BX to any other point X on the circle, and the radius CX.
$BC + AB = AC = CX$. By Theorem 7-7,
$BC + BX > CX$. Hence,
$BC + BX > BC + AB$ and
$BX > BA$.
Also $BX < BC + CX$,
or $BX < BC + CD = BD$.



*10. $(4000)^2 = (100)^2 + (4000 - x)^2$
$(4000 - x)^2 = 15{,}990{,}000.$
$4000 - x = 3{,}998.75.$
$x = 1.25$, approx.
The shaft will be about $1\frac{1}{4}$ miles deep.



460 11. $AY = AP$ and $AX = AP$, because tangent segments to a circle from an external point are congruent. Therefore, $AY = AX$.

*12. $AP^2 = 1(8 + 1) = 9$, by Theorem 13-13.

$AP = PX = XY = 3$, so $QX = 2$ and $XZ = 6$.

$3 \cdot AX = 2 \cdot 6$, by Theorem 13-14.

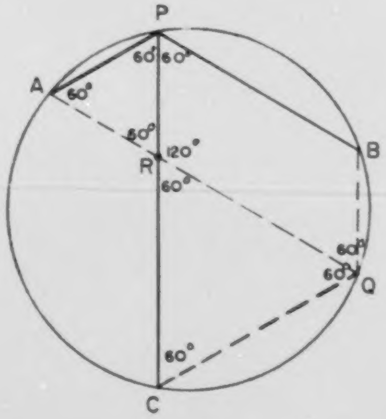
$AX = 4$.

*13. The angle measures can be determined as shown. Hence, Δ PAR and Δ QCR are equilateral triangles and PRQB is a parallelogram.

$PC = PR + RC = AR + RQ$.

But $AR = AP$ and $RQ = PB$. Hence,

$PC = AP + PB$.



Illustrative Test Items for Chapter 13

A. Indicate whether each of the following statements is true or false.

1. If a diameter of a circle bisects a chord of that circle which is not a diameter, then the diameter is perpendicular to the chord.

2. If a line bisects both the major and minor arcs of a given chord, then it also bisects that chord.

3. If two chords of a circle are not congruent, then the shorter chord is nearer the center of the circle.

4. If the measure of an angle inscribed in a circle is 90, then the measure of its intercepted arc is 45.

5. Any two angles which intercept the same arc of the same circle are congruent.

6. Two concentric circles have at least one point in common.

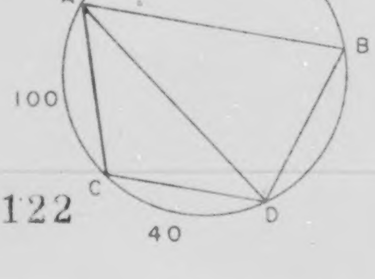
7. An angle inscribed in a semi-circle is a right angle.

8. If the interiors of two spheres each contain the same given point, then the spheres intersect in a circle.

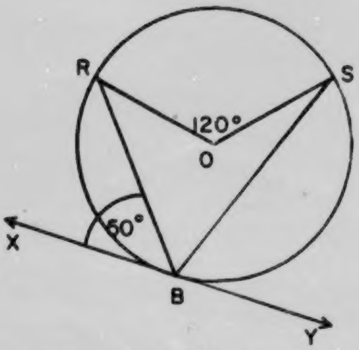
9. If two circles are tangent internally, then the segment joining their centers is shorter than the radius of either circle.

10. If two arcs, each of a different circle, have the same measure, then their chords are congruent.

B. 1. Given: $\overline{AB} \parallel \overline{CD}$ as shown, with m$\widehat{AC}$ = 100 and m$\widehat{CD}$ = 40.
Find: a. m$\angle$ B.
b. m$\angle$ C.
c. m$\angle$ DAB.



2. In the figure, $\overleftrightarrow{XY}$ is tangent to circle O at B. Find
   a. m∠ RBS.
   b. $m\overset{\frown}{BR}$.
   c. m∠ S.



C. 1. The mid-point of a chord 10 inches in length is 12 inches from the center of a circle. Find the length of the diameter.

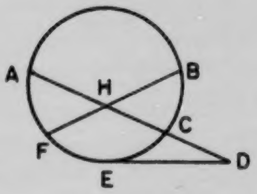
2. Two parallel chords of a circle each have length 16. The distance between them is 12. Find the radius of the circle.

3. Two concentric circles have radii of 6 and 2 respectively. Find the length of a chord of the larger circle which is tangent to the smaller circle.

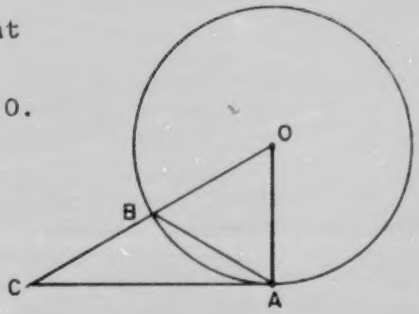
4. The distance from the mid-point of a chord 12 inches long to the mid-point of its minor arc is 4 inches. Find the radius of the circle.

5. In a circle, chords $\overline{AB}$ and $\overline{CD}$ intersect at E. AE = 18, EB = 8 and CE = 4. Find ED.

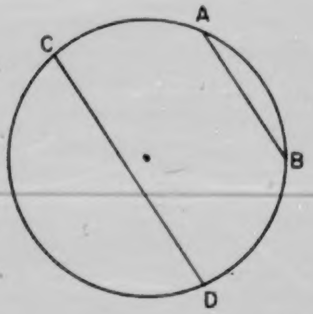
6. Given: Chord $\overline{BF}$ bisects chord $\overline{AC}$ at H. $\overleftrightarrow{DE}$ is a tangent. FH = 3, BH = 12 and CD = 3.
   Find: AC and DE.



D. 1. Given: $\overleftrightarrow{CA}$ is tangent to circle O at A.
Prove: $m\angle BAC = \frac{1}{2}m\angle O$.

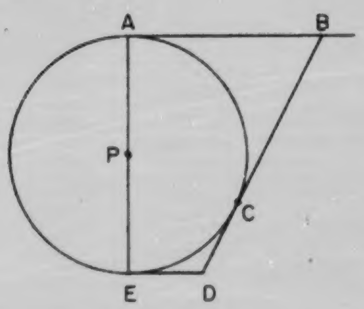


2. Given: $m\widehat{AC} = m\widehat{BD}$.
Prove: $\overline{AB} \parallel \overline{CD}$.



3. Prove that a parallelogram inscribed in a circle is a rectangle.

4. Given: Circle P with $\overleftrightarrow{AB}$, $\overleftrightarrow{BD}$, and $\overleftrightarrow{DE}$ tangent to the circle as shown.
Prove: AB + ED = BD.

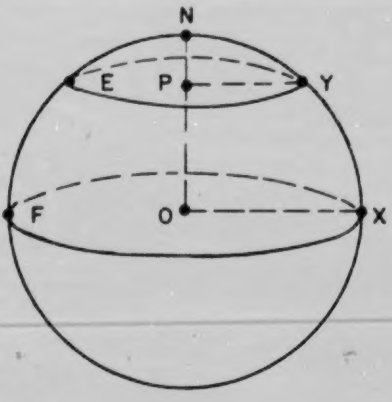


5. Given: Two circles are tangent at A and the smaller circle, P, passes through O, the center of the larger circle. The line of centers contains A.

Prove: The smaller circle bisects any chord of the larger circle that has A as an end-point.

6. $\overline{NO}$ is a radius of sphere O. At O, plane $F \perp \overline{NO}$. At P between N and O, plane $E \perp \overline{NO}$. $\overline{PY}$ and $\overline{OX}$ are coplanar radii of the circles in which E and F intersect sphere O.

If $m\widehat{NY} = \frac{1}{3}m\widehat{NX}$, explain why $PY = \frac{1}{2}OX$.



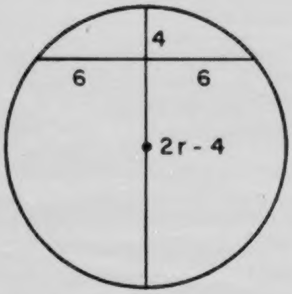
Answers

A. 1. True.
2. True.
3. False.
4. False.
5. False.
6. False.
7. True.
8. False.
9. False.
10. False.

B. 1. a. 70. b. 110 c. 50.
2. a. 60. b. 100 c. 20. (Use auxiliary segment $\overline{RS}$ or $\overline{OB}$.)

C. 1. 26 inches.
2. 10.
3. $8\sqrt{2}$ (from $2\sqrt{32}$).

4. Let r be the radius.
$36 = 4(2r - 4)$.
$52 = 8r$.
$6.5 = r$. The radius is 6.5 inches long.



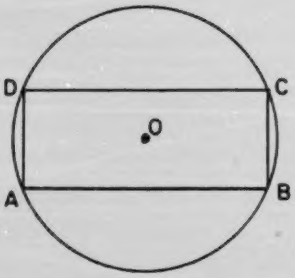
5. 36.

6. $AC = 12$, $DE = 3\sqrt{5}$.

D. 1. $m\angle BAC = \frac{1}{2}m\widehat{AB}$.
$m\angle O = m\widehat{AB}$.
Hence $m\angle BAC = \frac{1}{2}m\angle O$.

2. Draw $\overline{AD}$. Then $m\angle BAD = m\angle CDA$ since they intercept congruent arcs. $\overline{AB} \parallel \overline{CD}$, because of the congruent alternate interior angles formed.

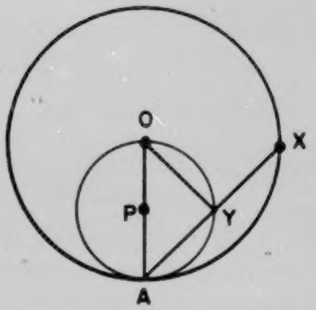
3. Given: ABCD is a parallelogram inscribed in circle O.
Prove: ABCD is a rectangle.



1.	$\angle D \cong \angle B$.	1.	Opposite angles of a parallelogram are congruent.
2.	$\widehat{ADC} \cong \widehat{ABC}$, and $\widehat{ABC}$ is a semi-circle.	2.	Arcs intercepted by congruent inscribed angles.
3.	$\angle D$ is a right angle.	3.	An angle inscribed in a semi-circle is a right angle.
4.	ABCD is a rectangle.	4.	Definition of rectangle and Theorem 9-23.

4. Since tangents to a circle from an external point are congruent, we have AB = BC and DE = DC. By addition, AB + DE = BD.

5. Let $\overline{AX}$ be a chord of circle O which intersects circle P at Y. Prove: AY = XY.



Consider $\overline{OY}$. $\angle$ AYO is a right angle, because it is inscribed in a semicircle. AY = XY because a line perpendicular to a chord and containing the center of the circle bisects the chord. (Since $\overline{OA}$ and $\overline{PA}$ are perpendicular to a common tangent at A, P must lie on $\overline{OA}$.)

6. Since $\overline{NO} \perp$ E at P, $\overline{NO} \perp \overline{PY}$, and Δ OPY is a right triangle. Since $\overline{NO} \perp$ F at O, $m\angle$ NOX = 90, and

$m\widehat{NY} = \frac{1}{3}m\widehat{NX} = 30$.

From properties of a 30 - 60 right triangle

PY = $\frac{1}{2}$OY. But OY = OX.

Therefore, PY = $\frac{1}{2}$OX.

Chapter 14

CHARACTERIZATION OF SETS. CONSTRUCTIONS.

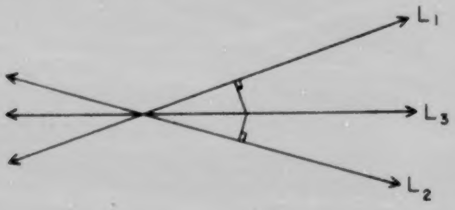
This chapter could be entitled Loci and Constructions. It deals with the traditional material of loci and ruler and compass constructions, and the treatment is mostly conventional. The only real innovation is the use of the term "characterization of a set" rather than "locus" as explained below.

The teacher may notice with relief or chagrin that the word locus does not occur in this chapter of the text. Its omission is deliberate. Conventional texts generally contain the phrase "locus of points" or "locus of a point". The phrase arose historically to mean (1) a description of the "location" of all points which satisfy a given condition or (2) the path of a point which "moves" so as to satisfy the condition. In each case the locus is a figure, that is, a set of points. Since we are already familiar with the term set, it seems undesirable to introduce a superfluous term which students often find confusing.

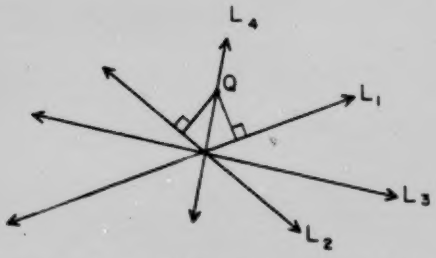
A more significant advantage, however, is that it allows us to concentrate on and develop the essential issue: to define each set by a common, or characteristic, property of its elements. We are concerned with defining, or characterizing, a set of points by means of a property which each point of the set must satisfy. Note that this point arises in other branches of mathematics. For example, in algebra we define the set of even integers by specifying a characteristic property (namely, divisibility by 2) satisfied by every even integer and by no other integer.

462 To summarize: We characterize a set by specifying a condition which is satisfied by all elements of the set, but no other elements; we call the condition a characterization of the set. To show that a certain set is characterized by a given condition, we must show (1) that each point of the set satisfies the given condition; and (2) each point satisfying the condition is a point of the set. Thus, we must prove (1) a theorem and (2) its converse. Sometimes it is convenient to prove (2) by the indirect method.

We mentioned above that in order to characterize a figure, we must prove a theorem and its converse. Consider the following example: Identify the set of points equidistant from two intersecting lines. Having drawn two intersecting lines L_1 and L_2 as below, a student might proceed to use the property that each point of the bisector of an angle is equidistant from the sides of the angle and conclude that L_3 is the required set of points.



His solution, however, is not correct, since he has found only a part of the required set. If he said that every point in this set was equidistant from the two intersecting lines, he would be correct, but if he were to try to establish that every point that satisfied the given condition was in this set, he would readily see his error. For there is a point Q, as in the figure below, that is equidistant from L_1 and L_2, but which does not lie in L_3. In fact there are many points which have this property, and we see that the set defined is not just one line, but two lines determined by the bisectors of the angles.



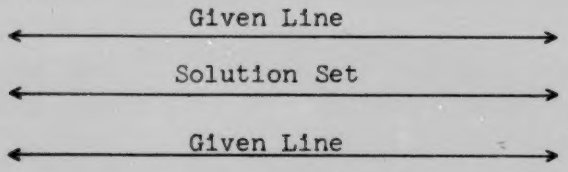
In Problem Set 14-1, the term cylindrical surface is used. The meaning should be intuitively clear to students and may be used accordingly.

Problem Set 14-1

463 1. The set of points is a sphere with center C and radius 3 inches.

2. The set of points is a circle in E with center C and radius 3 inches.

3. The set of points is the line in the plane E which is parallel to each of the given lines and equidistant from them.



4. Let O be the point of E which is the foot of the perpendicular from C to E (i.e. $\overline{OC}$ is 3 inches long).

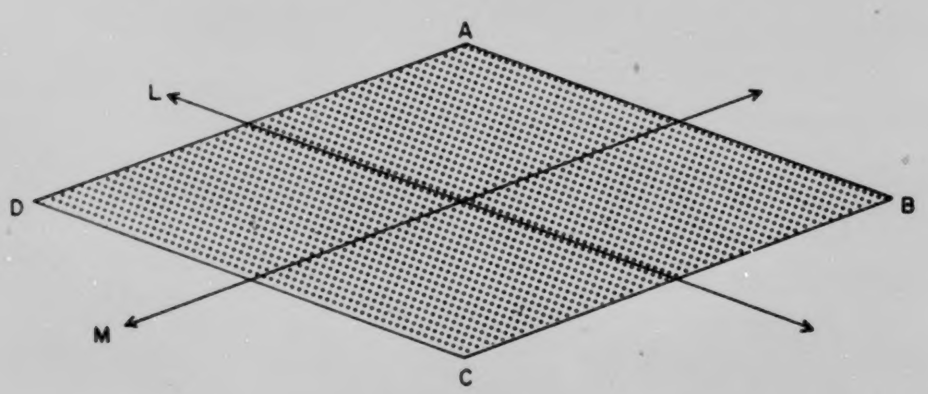
   a. The set of points is a circle with center O and radius 4 inches.

   b. The set consists of the single point O

463 c. There are no point of E 2 inches from C. Hence, the required set is the empty set.

5. a. There are four such points.

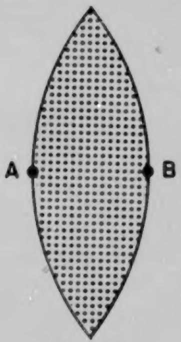
b.



$\overline{AB} \parallel L$; $\overline{CD} \parallel L$; $\overline{AD} \parallel M$; $\overline{BC} \parallel M$. The required set consists of the points of the parallelogram ABCD together with all interior points.

6. a. The set consists of two points, the third vertices of the two equilateral triangles which have $\overline{AB}$ as one side.

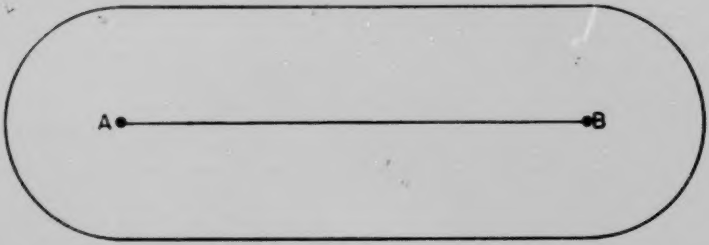
b. The solution set is the intersection of the two circular regions with centers A and B respectively and radii 4 feet.



463 c. The mid-point of $\overline{AB}$ is the only point of the set.

d. The empty set.

7. The set is the union of a pair of line segments parallel to and having the same length as $\overline{AB}$ and two semi-circles with radius 1 inch and centers A and B respectively, as shown.



Problem Set 14-2a

464 1. a. The sphere whose center is the given point and whose radius is the given distance.

b. The cylindrical surface with the given line as axis and the given distance as radius.

c. The two planes parallel to the given plane and at the given distance from it.

d. The four lines which are the intersections of the following sets of planes: two at the given distance from one of the given planes, two at the given distance from the other given plane.

464 e. The intersection of the two spheres having the given points as centers and the given distances as radii. This intersection may be a circle, one point, or the empty set.

f. A cylindrical surface (see b above) capped by two hemispheres.

2. a. The line which is the perpendicular bisector of the segment joining the two given points.

b. The line parallel to the given lines and midway between them.

c. The two lines which bisect the angles made by the given lines.

d. One point - the intersection of the perpendicular bisectors of two of the sides of the triangle determined by the given points.

465 3. a. The perpendicular bisecting plane of the segment joining the given points.

b. The perpendicular bisecting plane of a segment which is perpendicular at its end-points to the given lines.

c. The plane which is parallel to the given planes and midway between them.

d. Two planes which bisect the dihedral angles made by the given planes.

e. A conical surface composed of lines through the foot of the perpendicular and making 45° angles with the given line.

4. a. 1. true. 2. false.

b. 1. true. 2. false.

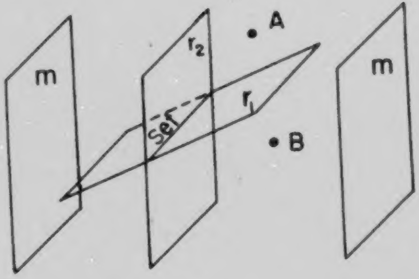
5. The pole should be placed at the point where the perpendicular bisectors of two sides of the triangle intersect.

465 6. The perpendicular bisecting plane of $\overline{AB}$, minus the mid-point of $\overline{AB}$.

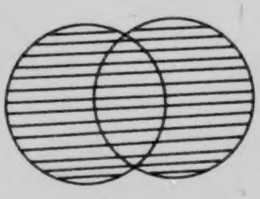
466 7. The point is the intersection of the perpendicular bisectors of two of the segments joining the pairs of points. If the points are collinear the two perpendicular bisectors will, of course, be parallel.

8. Points equidistant from two given points lie in a plane r_1. Points equidistant from two given parallel planes also lie in a plane r_2. In general, the intersection of two planes is a line, but if the two planes should be parallel, the intersection is the empty set or if the two planes should be equal the required set is a plane. In summary the set may be a line, a plane, or the empty set.

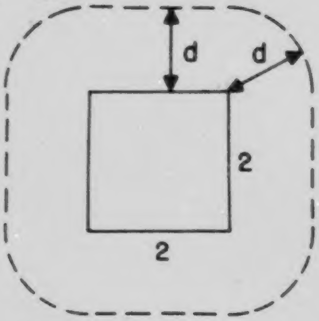
Given points A, B and parallel planes m and n.



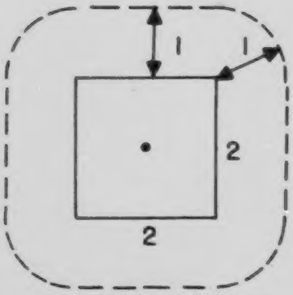
*9. The union of the interiors of two circles with 4 cm. radii and centers at the given points.



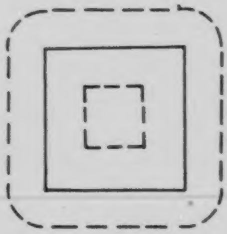
467 11.



d > 1

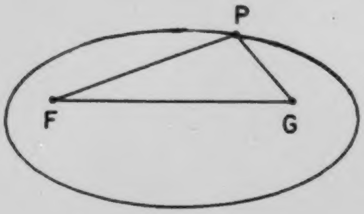


d = 1 (Center of square is part of the set)



d < 1

*12. Two pins are put in a drawing board, at F and G, and a loop of string of length 9 is placed around them and pulled taut by a pencil at P. As the pencil moves, always keeping the string taut, it describes a figure called an "ellipse".



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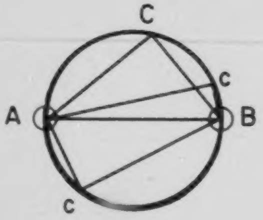
467 To justify Statement 1 we are assuming from the diagram that since D is in the interior of $\angle BAC$ so is P. (D is in the interior of $\angle BAC$ since $\overrightarrow{AD}$ is the bisector of $\angle BAC$.) This can be proved formally by using Theorem 6-5 and the definition of the interior of an angle.

In order to illustrate the precision with which we must define a set of points, the following problem might be presented to the class:

> Given two points A and B, what is the set of points C such that ΔABC is a right triangle?

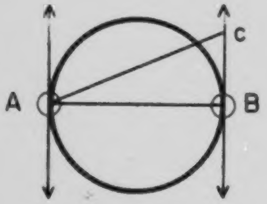
At first thought, one might consider that the angle inscribed in a semi-circle is a right angle and give the following as a picture of the set:

Note that points A and B are not in the set.



However, the problem did not say, "What is the set of points C such that ΔABC is a right triangle with right angle at C." The right angle might equally well be at A or at B, and we have to draw the set like this:

Note again that points A and B are not in the set.



The set consists of all points on a circle with diameter $\overline{AB}$ and also all points on the lines perpendicular to this diameter at A and B excluding the points A and B.

469 In Theorem 14-2 we are referring, of course, to the perpendicular bisectors of the sides in the plane of the triangle.

Theorem 14-2 will be used later to circumscribe a circle about a triangle. The construction is a direct consequence of the theorem. Since the point of concurrency is the center of the circumscribed circle, it is called the circumcenter of the triangle.

In the proof of Theorem 14-2 we can answer the question "Why?", as follows. Suppose $L_1 \parallel L_2$. We know $\overleftrightarrow{AB} \perp L_1$ and $\overleftrightarrow{AC} \perp L_2$. Hence $\overleftrightarrow{AB} \perp L_2$. Thus the two lines $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$ are perpendicular to L_2, and must be parallel.

Proofs of the Corollaries

470 Corollary 14-2-1. There is one and only one circle through three non-collinear points.

Since the existence and uniqueness of a point equidistant from the three vertices of a triangle is proved in Theorem 14-2, the center and radius of a circle containing any three non-collinear points are uniquely determined.

Corollary 14-2-1. Two distinct circles can intersect in at most two points.

Theorem 13-2 rules out the possibility of more than two collinear points and Corollary 14-2-1 rules out the possibility of three, or more, non-collinear points.

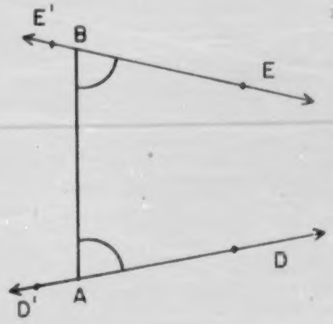
470 In the proof of Theorem 14-3, L_1 is perpendicular to $\overleftrightarrow{DE}$ because $L_1 \perp \overleftrightarrow{BC}$ and $\overleftrightarrow{BC} \parallel \overleftrightarrow{DE}$.

The point of concurrency of the altitudes of a triangle is called the orthocenter.

We have shown in Theorem 9-27 that the medians of a triangle are concurrent at a point, called the centroid of the triangle.

It is interesting to note that in a given triangle, the orthocenter, circumcenter and the centroid are collinear. This leads to an interesting problem. If we draw the segment between the orthocenter and the circumcenter and find its mid-point, then using this point as center and the distance from this point to the mid-points of the sides of the triangle as a radius and draw the circle defined by these conditions, we get what is called the Nine-point Circle. This circle has the following properties: It passes through the mid-points of the sides, it passes through the feet of the three altitudes of the triangle, and it passes through the mid-points of the segments joining the orthocenter (point of intersection of the altitudes) to the vertices.

471 For complete rigor in the proof of Theorem 14-4, one should first prove that $\overrightarrow{AD}$ and $\overrightarrow{BE}$ really do intersect. The proof is as follows: Since $m\angle A + m\angle B + m\angle C = 180$, and $m\angle ABE < m\angle B$, and $m\angle BAD < \angle A$, then we have
$m\angle ABE + m\angle BAD < 180$.
Now $\overleftrightarrow{BE}$ and $\overleftrightarrow{AD}$ are not parallel, since otherwise we would have $m\angle ABE + m\angle BAD = 180$. (We are using the fact that E and D are on the same side of $\overleftrightarrow{AB}$ to ensure that $\angle ABE$ and $\angle BAD$ are a pair of interior angles on the same side of the transversal $\overleftrightarrow{AB}$.) Thus $\overleftrightarrow{BE}$ and $\overleftrightarrow{AD}$ intersect.



Let $\overrightarrow{BE'}$ and $\overrightarrow{AD'}$ be the rays opposite to $\overrightarrow{BE}$ and $\overrightarrow{AD}$. Then one of the four cases must hold:

(1) $\overrightarrow{BE'}$ intersects $\overrightarrow{AD'}$. This is impossible since if their point of intersection were T, the triangle TAB would have two angles the sum of whose measures was more than 180.

(2) $\overrightarrow{BE'}$ intersects $\overrightarrow{AD}$. This is impossible, since the rays lie on opposite sides of $\overleftrightarrow{AB}$.

(3) $\overrightarrow{BE}$ intersects $\overrightarrow{AD'}$. This is impossible for the same reason as (2).

(4) $\overrightarrow{BE}$ intersects $\overrightarrow{AD}$. Being the only possibility left, this must be true.

Notice that we have used no special property of bisectors, merely the fact that $\overrightarrow{BE}$ and $\overrightarrow{AD}$ (excluding B and A) are in the interiors of $\angle B$ and $\angle A$.

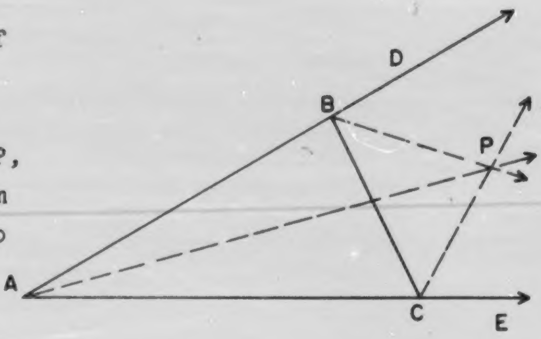
Theorem 14-4 will be used to inscribe a circle in a triangle. We can see that the point of intersection is equidistant from the sides of the triangle, and a circle with this point as center and the distance from this point to a side as radius, will have the sides of the triangle as tangents. This point of concurrency is called the incenter of the triangle.

Problem Set 14-2b

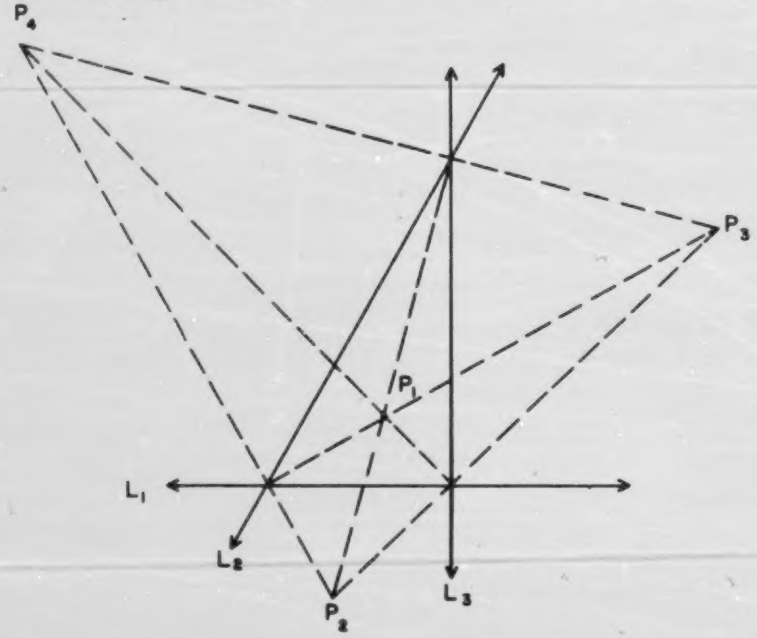
472 1. The point is the intersection of $\overleftrightarrow{PQ}$ and the bisector of $\angle B$.

2. The fountain should be placed at the intersection of the bisector of $\angle B$ and the perpendicular bisector of $\overline{DC}$.

3. The proof is almost identical with that of Theorem 14-4: If the bisectors of $\angle BAC$ and $\angle DBC$ meet at P, P is equidistant from $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$, and also from $\overleftrightarrow{BD}$ and $\overleftrightarrow{BC}$. But $\overleftrightarrow{AB} = \overleftrightarrow{BD}$, hence, P is equidistant from $\overleftrightarrow{CE}$ and $\overleftrightarrow{BC}$ and lies on the bisector of $\angle BCE$.



472 4. This follows by applying Theorem 14-4 and Problem 3 to the bisectors of the interior and exterior angles of the triangle as shown.



473 5. Let m be the radius of any circle with center M and n be the radius of any circle with center N. Then the situations are:

a. $m + n < MN$.

b. $m > MN + n$ or $n > MN + m$.

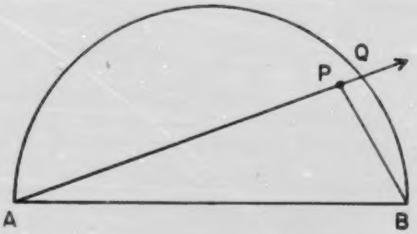
6. The angle bisectors are not necessarily concurrent. They are concurrent for a square or a rhombus. In general, they are concurrent if and only if there exists a circle tangent to each of the sides of the quadrilateral.

7. Each of the six segments is a chord of the circle. Hence, each perpendicular bisector passes through the center of the circle.

473 8. The required set is the circle with the segment as diameter, but with the end-points of the segment omitted.

If P is in this set, then $\angle$ APB is a right triangle by Corollary 13-7-1.

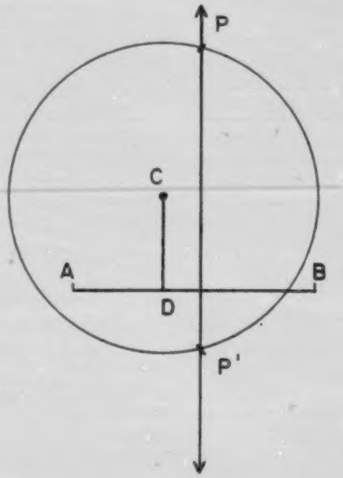
If $\angle$ APB is a right angle, let $\overrightarrow{AP}$ intersect the circle in Q. Then $\angle$ AQB is a right angle by Corollary 13-7-1, and hence, Q = P by Theorem 6-3. Therefore P lies on the circle, but $P \neq A$ and $P \neq B$.



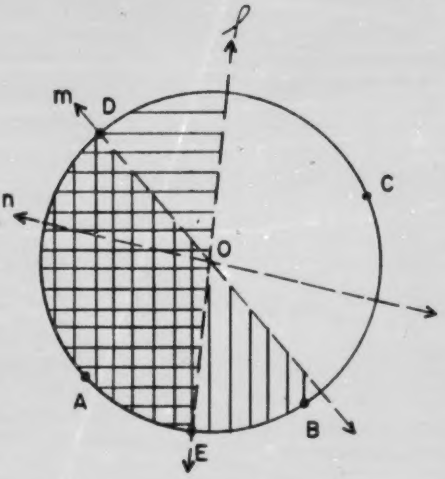
Problem Set 14-3

474 1. There will be two points P, the intersections of the circle with center A and radius 4, and the circle with center B and radius 5.

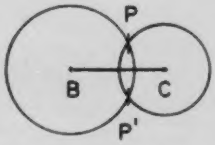
2. The two points P, P', are the intersections of the perpendicular bisector of $\overline{AB}$ and the circle whose center is C and whose radius is 5.



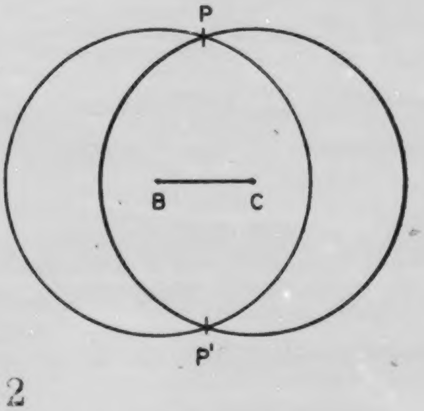
474 *3. ℓ, m, n are the $\perp$ bisectors of $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ respectively. Each passes through the center O of the circle. Thus the points interior to the circle and to the left of ℓ (shaded horizontally) are nearer to A than to B. Similarly the points inside the semi-circular region shaded vertically are nearer to A than to C. The required set is the intersection of the interiors of these two semi-circular regions (the interior of the sector ODAE).



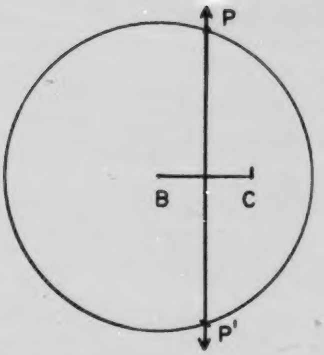
475 4. a. Two points, the intersections of the circle with center B and radius 4, and the circle with center C and radius 3.



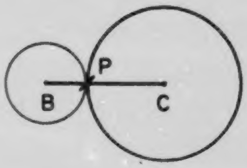
b. Two points, the intersections of circles with centers B and C and radius 10.



475 c. Two points, the intersections of the circle with center B and radius 10, and the perpendicular bisector of $\overline{BC}$.



d. One point, the intersection of $\overline{BC}$ and the circle with center B and radius 2, and the circle with center C and radius 4.



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475 The inclusion of some compass and straight-edge constructions in the text is a luxury, a concession to the interest this traditional topic has always generated in geometry classes. Under ruler and protractor postulates the restriction to compass and unmarked straight-edge is quite artificial. For example, to divide a segment into seven congruent segments we need only to divide its length by seven and plot the appropriate points on the segment. An angle can be divided up by a similar process using a protractor. Certainly one of the quickest ways to construct a perpendicular to a line is to use a protractor to construct an angle of 90°.

The main reason for this bow to tradition, then, is to attempt to capture the interest which arises from the challenge that constructions provide. Historically, compass and straightedge constructions have been tremendously important in stimulating significant advances in mathematics, as in the theory of higher degree equations or in proving that π is a transcendental number. We hope that your students will likewise enjoy and benefit from the many challenges found in the theorems and problems of these sections.

476 The absence of Theorem 14-5 in Euclid's Elements is one of the reasons why present-day geometers state that the postulate system of Euclid is incomplete. For a more complete discussion of the need for this theorem see Studies II.

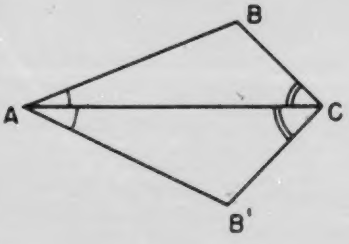
477 Notice that for every construction, the text gives a proof. When the students do some of the constructions for themselves, some of these should be accompanied by a proof that the construction is correct. A careful analysis of a construction problem will yield a proof with just a little more work than doing the construction.

479 Notice how the Two-Circle-Theorem is used to establish that the two circles in this construction theorem do actually intersect.

Problem Set 14-5a

480 1. Part d is not possible.

2.



3. $\angle C$. $\frac{BC}{HQ} = \frac{AC}{MQ}$.

4. a. If the length of the given segment $\overline{AB}$ is c, draw the circles with center A and radius c, center B and radius c. Since $c + c > c$, these circles intersect at C and C', say, and Δ ABC and Δ ABC' are equilateral.

480 b. If c is the length of the given base $\overline{AB}$ and r is the length of the side, then the two circles with centers A and B and radius r will intersect at C, C', say, if and only if, $r > \frac{c}{2}$, and Δ ABC and Δ ABC' will be isosceles with base $\overline{AB}$.

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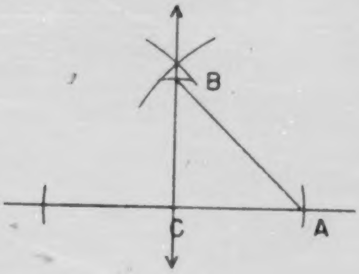
481 In Construction 14-8, the condition that r should satisfy to insure the intersection of the circular arcs in two points, is that r must be greater than $\frac{1}{2}$ the length of the given segment. In this particular problem, $r > \frac{1}{2}AB$. A value of r that is sure to work is r equal to the length of the given segment; in this problem $r = AB$ will always work.

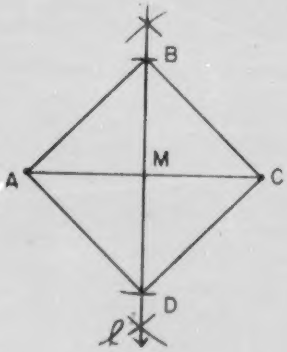
482 Notice that Construction 14-9 works just as well if P is on L.

Problem Set 14-5b

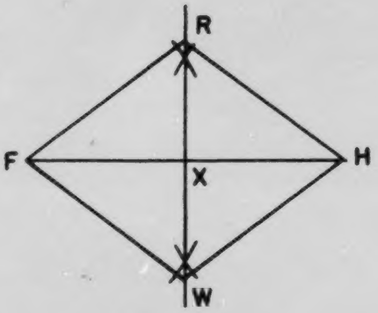
483 1. Construct $\overleftrightarrow{BC} \perp \overline{AC}$. Make $\overline{BC} \cong \overline{AC}$. Δ ABC is the required triangle.



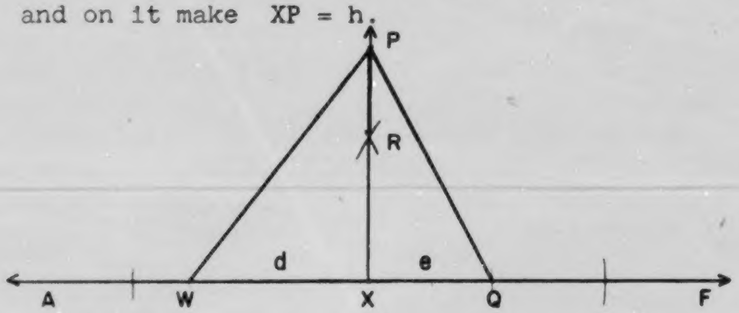
2. Construct the perpendicular bisector ℓ of $\overline{AC}$, meeting $\overline{AC}$ in M. Mark off $\overline{MB}$, $\overline{MD}$ on ℓ, each congruent to $\overline{AM}$. ABCD is the required square.



483 3. Make $\overline{FH} \cong \overline{AB}$. Construct the perpendicular bisector of $\overline{FH}$. Make $\overline{EQ} \cong \overline{CD}$. Bisect $\overline{EQ}$. Make $XR = XW = \frac{1}{2}EQ$. FWHR is the required rhombus.



4. On $\overleftrightarrow{AF}$ as a "working line", make $XW = d$ and $XQ = e$. $\overline{QW}$ is the base of our triangle. Construct $\overrightarrow{XR} \perp \overleftrightarrow{AF}$ and on it make $XP = h$.



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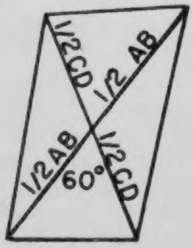
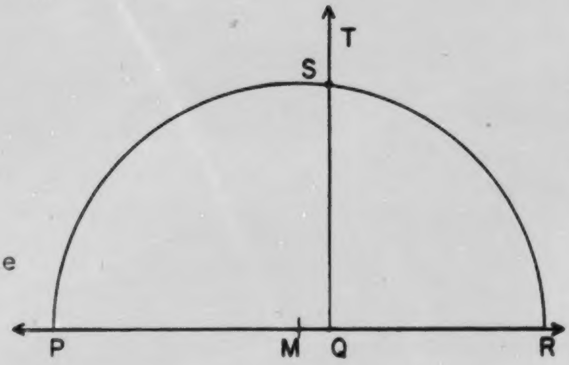


6. $PQ = AB$, $QR = CD$. M is the mid-point of $\overline{PR}$. $\overrightarrow{QT} \perp \overleftrightarrow{PR}$. $\overrightarrow{QT}$ meets semi-circle at S. QS is the geometric mean of AB and CD.



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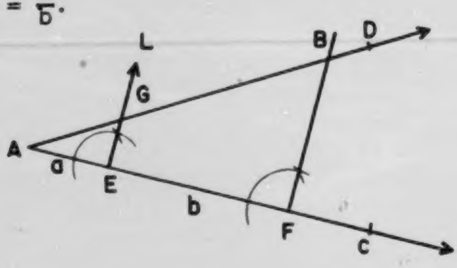
484 Other ways to construct a line parallel to a given line through an external point are (1) construct corresponding angles congruent and (2) construct a line perpendicular to a line through the given point perpendicular to the given line.

485 On the basis of a construction very similar to 14-11 it is possible to divide the length of a given segment in a given ratio. Given a segment $\overline{AB}$, we want to divide AB into two segments such that the lengths of these segments will be in some given ratio, say $\frac{a}{b}$. The construction is as follows:

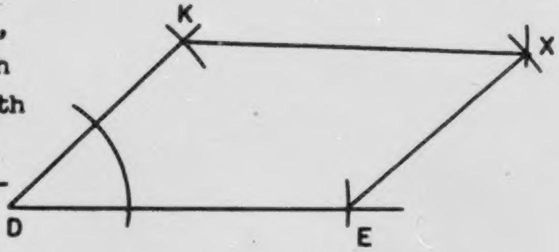
Starting at A draw any ray $\overrightarrow{AD}$, and a ray $\overrightarrow{AC}$ not collinear with ray $\overrightarrow{AD}$. On $\overrightarrow{AD}$ mark off AB and on $\overrightarrow{AC}$ mark off AE = a and EF = b. Draw $\overline{BF}$, and through E construct a line L parallel to $\overleftrightarrow{BF}$ intersecting $\overrightarrow{AD}$ at G. Then $\frac{AG}{GB} = \frac{a}{b}$.



Proof: Since we have in Δ ABF, $\overleftrightarrow{EG}$ parallel to $\overleftrightarrow{BF}$ and intersecting $\overline{AF}$ and $\overline{AB}$, then it follows from Theorem 12-1 that $\frac{AG}{GB} = \frac{AE}{EF}$, hence, $\frac{AG}{GB} = \frac{a}{b}$.

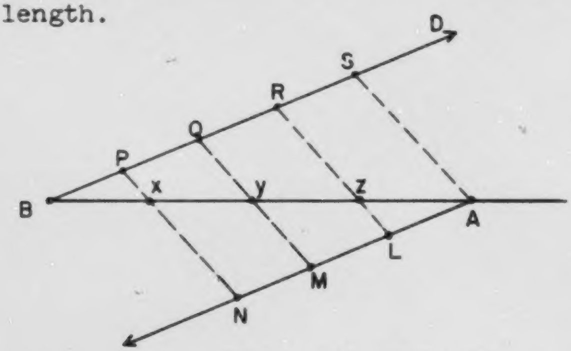
Problem Set 14-5c

485 1. Make $\overline{DE} \cong \overline{AB}$. Make $\angle D \cong \angle Q$ and $\overline{DK} \cong \overline{FH}$. Using E as center and the length FH as radius, strike an arc and with K as center and length AB as radius, strike another arc intersecting the first at X on the opposite side of $\overleftrightarrow{KE}$ from D. DEXK is the required parallelogram. (If both pairs of opposite sides of a quadrilateral are congruent, it is a parallelogram.)



486 2. Using OA as radius and O as center, construct an arc as shown. Count the number of small arcs (9 in this example) and draw a radius from O to the intersection of the arc and the (n + 1)th line (10th in this case). The radius $\overline{OB}$ congruent to the original segment, will be divided by the lines of the paper into congruent segments, which may be marked off on $\overline{OA}$. We assume that the lines of the paper are parallel and that they intercept congruent segments on one transversal (the margin of the sheet of paper). See Theorem 9-26.

486 3. Corresponding segments on $\overleftrightarrow{BD}$, $\overleftrightarrow{AC}$ are parallel and of equal length.



Hence, the segments $\overline{PN}$, $\overline{QM}$, $\overline{RL}$, $\overline{SA}$ are parallel.

Hence, $\frac{BX}{XY} = \frac{BP}{PQ} = 1$ and $BX = XY$. Similarly, $XY = YZ = ZA$.

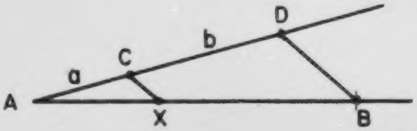
487 4. Divide $\overline{AB}$ into three congruent segments. Construct an equilateral triangle with one of these segments as side.

5. Divide $\overline{AB}$ into five congruent segments. Use one of them as the base.

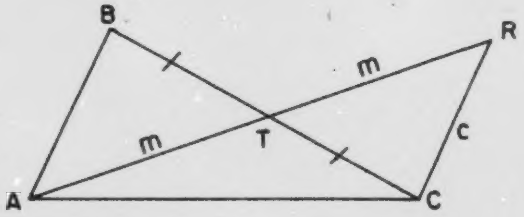
6. In effect we have here, "If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram", and we know that the opposite sides of a parallelogram are parallel.

   Alternate proof: Use S.A.S. and alternate interior angles.

7. On an arbitrary ray through A lay-off segments $\overline{AC}$ and $\overline{CD}$, with C between A and D, of lengths a and b. Through C draw $\overline{CX} \parallel \overline{DB}$. Δ ACX and Δ ADB are similar (A.A.) and have $\frac{AX}{XB} = \frac{a}{b}$.



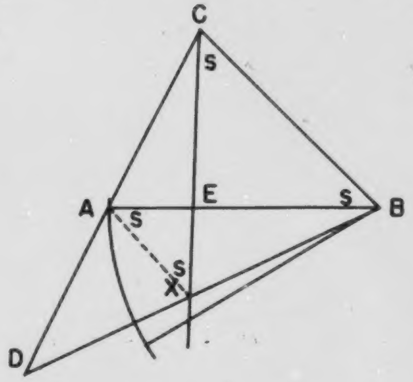
487 *8. Construct a triangle ARC with $AC = b$, $CR = c$, $AR = 2m$.



Bisect $\overline{AR}$ at T. On $\overleftrightarrow{CT}$ take B so that $CT = TB$. Then Δ ABC is the required triangle for $\Delta ABT \cong \Delta RCT$ by S.A.S., so $AB = CR = c$. Clearly, $\overline{AT}$ is the median and $AT = m$ by construction.

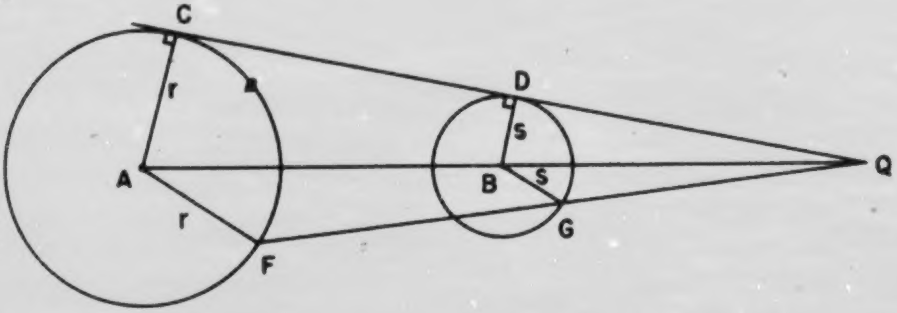
488 *9. Construct $AB = x$ and divide it into three congruent segments. At E (one of the trisection points) construct $\overleftrightarrow{CE} \perp \overline{AB}$. Make $EX = AE$, $CE = BE$. Δ DBC is the required triangle.

To prove that $\overline{BA}$ and $\overline{CX}$ are medians, draw $\overline{AX}$. Now, $\triangle$s EAX and ECB are isosceles triangles with congruent vertex angles and so angles s are all congruent. Then $\overline{AX} \parallel \overline{CB}$ and $\Delta EAX \sim \Delta EBC$ with $\frac{AX}{BC} = \frac{1}{2}$. Also $\Delta DAX \sim \Delta DCB$ and $\frac{DA}{DC} = \frac{1}{2}$, so A is the mid-point. If $\overleftrightarrow{CA}$ and $\overleftrightarrow{BX}$ were parallel, AX would have to equal CB. This we have shown is not true, so $\overleftrightarrow{CA}$ and $\overleftrightarrow{BX}$ must intersect.



488 *10. Analysis of problem: The common tangent $\overrightarrow{LN}$ meets m at L, and $LK = LN = LM$, so L is the mid-point of $\overline{KM}$. $\overline{PN} \perp \overline{LN}$ and $\overline{PM} \perp m$. Now proceed as follows: Bisect $\overline{KM}$; let L be the mid-point. With center L and radius LK construct an arc intersecting circle C at N. Construct $\overleftrightarrow{MS} \perp m$ and $\overleftrightarrow{NR} \perp \overleftrightarrow{LN}$, intersecting in P. Then $PM = PN$ and the required circle has center P and radius PM.

489 *11.



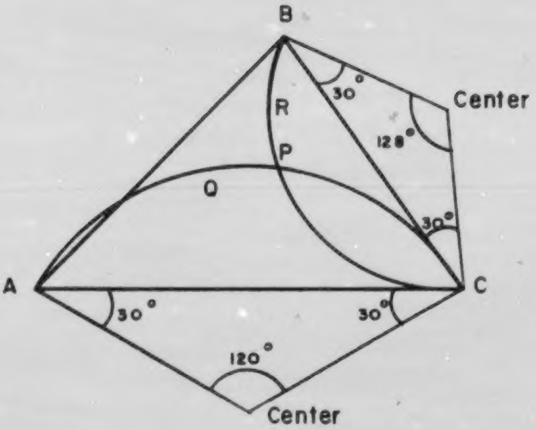
The problem will be solved if we can find Q, the intersection of the common external tangent and the line determined by the centers. In the figure, △s QDB and QCA are similar, being right triangles with a common acute angle.
Therefore,

$$\frac{AQ}{BQ} = \frac{AC}{BD} = \frac{r}{s}.$$

We can find Q by drawing a ray $\overrightarrow{AF}$ making a convenient angle with $\overrightarrow{AB}$, then drawing the ray $\overrightarrow{BG}$ parallel to $\overrightarrow{AF}$ and on the same side of $\overrightarrow{AB}$. Q is determined as the intersection of $\overrightarrow{FG}$ and $\overrightarrow{AB}$, since triangles AFQ and BGQ are similar, and

$$\frac{AQ}{BQ} = \frac{AF}{BG} = \frac{r}{s}, \text{ as desired.}$$

489 *12.



Let $\overset{\frown}{AQC}$ be an arc of 120°. Then $m\angle AQC = 120$ for any position of Q on the arc. Similarly, let $\overset{\frown}{BRC}$ be an arc of 120°. Hence, if P is the point of intersection (other than C) of the two arcs, we have $m\angle APC = m\angle BPC = 120$. It follows that $m\angle APB = 120$. (A complete analysis of this problem, including the case in which one angle has measure ≥ 120, is very complicated.)

*13. By A.A., $\triangle BPM \sim \triangle DPN$ and $\triangle MPC \sim \triangle NPA$ so that

$$\frac{MB}{ND} = \frac{MP}{NP} \quad \text{and} \quad \frac{MC}{NA} = \frac{MP}{NP}.$$

Hence, $$\frac{MB}{ND} = \frac{MC}{NA} \quad \text{or} \quad \frac{MB}{MC} = \frac{ND}{NA}.$$

By A.A., $\triangle QBM \sim \triangle QAN$ and $\triangle QCM \sim \triangle QDN$ so that

$$\frac{MB}{NA} = \frac{MQ}{NQ} \quad \text{and} \quad \frac{MC}{ND} = \frac{MQ}{NQ}.$$

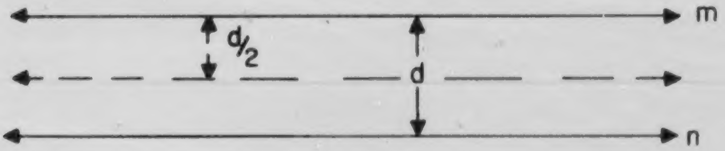
Hence, $$\frac{MB}{NA} = \frac{MC}{ND} \quad \text{or} \quad \frac{MC}{MB} = \frac{ND}{NA}.$$

Thus the ratios $\frac{MB}{MC}$ and $\frac{MC}{MB}$ are each equal to $\frac{ND}{NA}$.

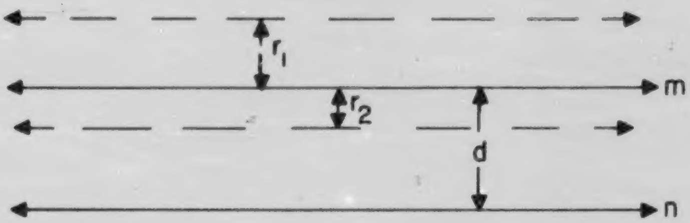
Therefore $$\frac{MB}{MC} = \frac{MC}{MB}, \quad MC^2 = MB^2,$$

and $$MC = MB.$$

489 *14. If $k = 1$, the required set is the line parallel to m and n and at a distance $\frac{d}{2}$ from each.



If $k < 1$, the required set is the union of two lines, one such that



$\frac{r_1}{r_1 + d} = k$, or $r_1 = \frac{kd}{1 - k}$, and the other such that

$\frac{r_2}{d - r_2} = k$, or $r_2 = \frac{kd}{1 + k}$.

If $k > 1$, interchange the roles of m and n.

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496 To construct the number in the text requires 15 steps. To verify this, we must know what we mean by a step. A step is one operation of addition, subtraction, multiplication or division. In this example we start with the integers 1,2, 3,4,5,7,9,10,17,37, and 47. We construct $\frac{5}{2}$ by one step, divide 5 by 2. In like manner, to construct $\frac{17}{37}$, $\frac{3}{4}$, $\frac{7}{3}$, $\frac{1}{7}$, $\frac{3}{5}$, $\frac{9}{10}$ and $\frac{37}{47}$ requires one step each; hence, to get these numbers we require 8 steps. To construct the numbers $a = \frac{5}{2} - \frac{17}{37}$, $b = \frac{3}{4} + \frac{7}{3}$, $c = \frac{1}{7} + \frac{3}{5}$, and $d = \frac{9}{10} - \frac{37}{47}$ requires 4 steps, two additions and 2 subtractions. We have now used 12 steps and have arrived at four numbers,

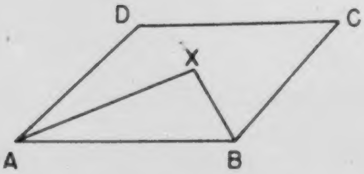
a, b, c, and d. Now perform two divisions (2 steps) and get the numbers $\frac{a}{b}$ and $\frac{c}{d}$. Now we have used up 14 steps. Finally make one division (1 step) $\frac{a}{b} \div \frac{c}{d}$, and we have now constructed the number given in the text in 15 steps. The number is $\frac{3,681,962}{507,899}$.

497 In the figure for the trisection problem, it is interesting to see that as m∠ ABC increases the marked point P moves in a very limited range on the ray opposite to $\overrightarrow{BA}$. This range is $\sqrt{2}r \leq PB < 2r$. If C coincides with A then PB = 2r and we do not have an angle to trisect. As m∠ ABC increases C and Q approach coincidence. When they coincide the ruler cuts the circle in only one point Q, and $\overline{BQ} \perp \overline{PQ}$ and $BP = \sqrt{2}r$. The largest angle we can trisect by this method is a 135° angle. The trisection of any obtuse angle can be reduced to the trisection of an acute angle.

Problem Set 14-7

499 1. Since for each parallelogram ∠ A and ∠ B are supplementary,

$m\angle \frac{1}{2}A + m\angle \frac{1}{2}B = 90$.



This means that the bisectors must be perpendicular to each other. Then the required set will be the circle whose diameter is $\overline{AB}$, except for points A and B.

2. a. Bisect a 90° angle.
   b. Bisect a 60° angle (one angle of an equilateral triangle).
   c. Bisect a 45° angle. (See a.)

499 d. $90^\circ + 45^\circ$, or $180^\circ - 45^\circ$.

e. $60^\circ + 60^\circ$, or $180^\circ - 60^\circ$.

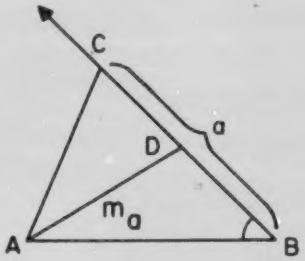
f. $30^\circ + 45^\circ$, or $90^\circ - 15^\circ$.

g. $60^\circ + 45^\circ$, or $90^\circ + 15^\circ$.

h. $22\frac{1}{2}^\circ$ is half of 45°, and $67\frac{1}{2}^\circ = 45^\circ + 22\frac{1}{2}^\circ$.

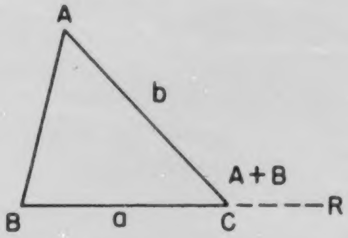
500 3. In the discussion that follows, each figure is merely a sketch of the completed figure.

a. Construct $\angle B$ congruent to the given angle. Make $BC = a$. Find the mid-point D of $\overline{BC}$. Use D as center and m_a as radius to intersect $\overrightarrow{BA}$ at A.

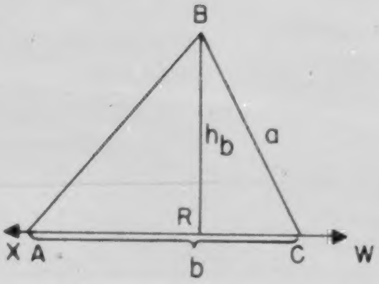


There are cases in which the construction is impossible and cases in which there are two solutions.

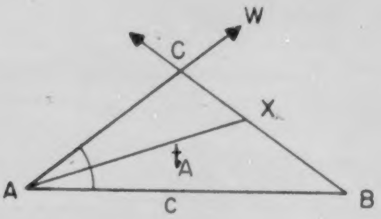
b. Construct $\angle ACR \cong \angle X$. Then $\angle ACB$ is the third angle of the triangle. Make $CA = b$ and $CB = a$.



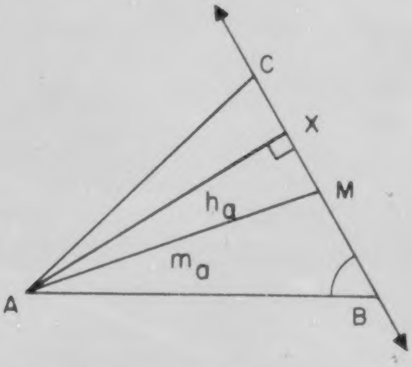
500 c. Construct a segment $\overline{RB}$ perpendicular to a "working line", $\overleftrightarrow{XW}$, at any convenient point and make $RB = h_b$. Using B as center and a as radius, construct an arc intersecting $\overleftrightarrow{XW}$ at C. Using C as center and b as radius construct an arc intersecting $\overleftrightarrow{XW}$ at A. (Two solutions in general, depending upon where A is taken, on the ray $\overrightarrow{CR}$ or on the opposite ray.)



d. Construct $\angle$ WAB congruent to $\angle$ A. Construct its bisector, $\overline{AX}$. Make AB = c. Connect B with X. The point at which $\overleftrightarrow{BX}$ meets $\overleftrightarrow{AW}$ is C. (There may be no solution, in case $\overleftrightarrow{BX} \parallel \overleftrightarrow{AW}$.)



e. Since we are given $\angle$ B and since $m\angle AXB = 90$, we can construct $\angle$ XAB. Then construct Δ ABX by constructing $\overline{AX}$ (of length h_a) $\perp \overline{BC}$, and $\angle$ XAB. Using A as center and m_a as radius, find M, then make MC = MB.



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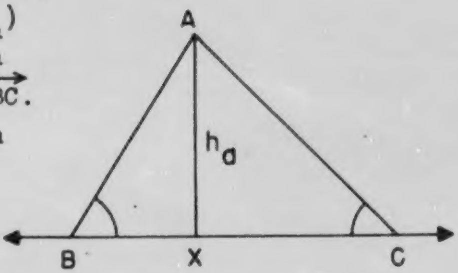
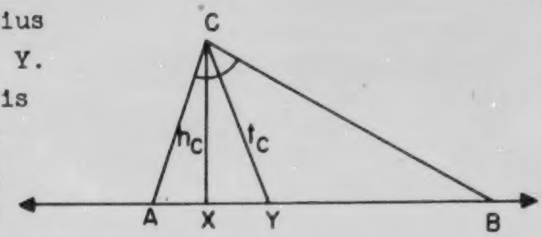
400

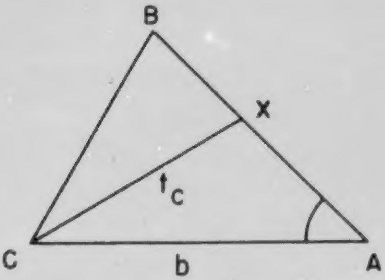
500 f. Start by constructing $\overline{AX}$ (of length h_a) perpendicular to a "working line", $\overleftrightarrow{BC}$. Since we are given $\angle B$ and since $m\angle BXA = 90$, $m\angle BAX$ can be easily constructed. Similarly $\angle CAX$ can be constructed. Construct these two angles at A.



g. Construct $\overline{CX}$ (of length h_c) perpendicular to $\overleftrightarrow{AB}$. Use C as center and t_c as radius to cut $\overleftrightarrow{AB}$ at Y. Now since $\overline{CY}$ is the bisector of $\angle C$, construct on each side of $\overline{CY}$ and angle whose measure is $\frac{1}{2} m\angle C$.



h. Construct an angle congruent to $\angle A$ and make $AC = b$. Using C as center and t_C as radius, find X. We now have $\angle XCA$ of measure $\frac{1}{2} m\angle C$. Construct $\angle XCB \cong \angle XCA$.



500 4. Δ BPM $\sim \Delta$ DPA, by A.A.A., and so

$$\frac{BP}{DP} = \frac{BM}{DA} = \frac{1}{2}.$$

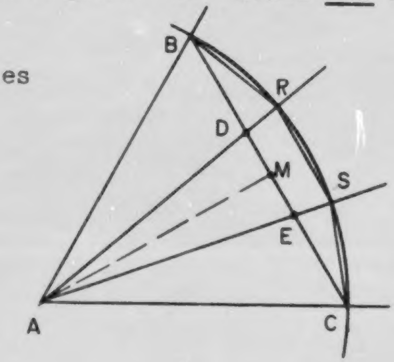
Hence, $BP = \frac{1}{3} BD$.

A similar argument shows $DQ = \frac{1}{3} DB$, so that P and Q are the trisection points of $\overline{BD}$.

In the right triangle ABM, the ratio $\frac{AB}{BM} = 2$.

If $m\angle BAM = 30$, this ratio would have to be $\sqrt{3}$, and hence, $m\angle BAM \neq \frac{1}{3} \cdot 90^{\circ}$.

Hence the trisection of the segment $\overline{BD}$ would not lead to trisecting $\angle$ BAD.



501 5. a. Definition of isosceles triangle.

b. D and E will be inside the circle, because AD and AE are each less than the radius. This can be shown by considering a segment joining A to the mid-point M of $\overline{BC}$. $\overline{AM} \perp \overline{BC}$ and D and E are nearer M than B and C are.

If $\overline{RE}$ is drawn, area Δ BRD = area Δ EDR, hence, area DRSE > area Δ BRD, and, by addition, area Δ ARS > area Δ ARB. But if $\angle$ BAC were trisected, we would have area Δ ARS = area Δ ARB.

6. Let $\overleftrightarrow{QD}$ meet $\overline{BA}$ at G, and drop $\overline{AH} \perp \overrightarrow{QP}$. Then

$$\Delta QHA \cong \Delta QGA \cong \Delta QGB,$$

from which the desired result follows. Notice that $\overrightarrow{QA}$ and $\overrightarrow{QG}$ are trisectors of $\angle$ PQR.

Review Problems

503 1. 3, 4, 5, 6, 7, 8, 9.

2. Divide $\overline{AB}$ into 4 congruent segments. Bisect a 90° angle. Construct the rhombus using a 45° angle and $\frac{1}{4}\overline{AB}$ for each side of the rhombus.

3. a. Construct the circle on $\overline{AB}$ as diameter. The circle minus A and B is the set of points P.

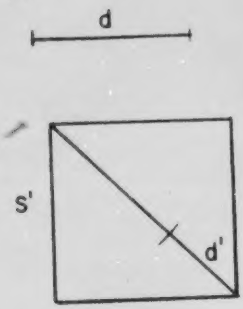
   b. See the solution of Problem 8 of Problem Set 14-2b.

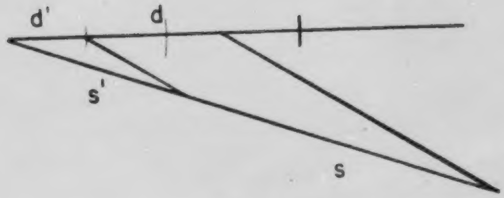
4. The set is the intersection of two parallel lines (each at distance d from L) and a circle (with radius r and center P). This intersection may be the empty set or 1, 2, 3 or 4 points.

5. Examples of such quadrilaterals are rectangles and isosceles trapezoids. More generally, if a quadrilateral has this property, then the point of concurrency is equidistant from each vertex, hence, the circle with the point of concurrency as center and the distance to each vertex as radius passes through each vertex. Conversely, any quadrilateral whose vertices lie on some circle has the property that the perpendicular bisectors of the sides are concurrent, so that a quadrilateral has this property if and only if, there is a circle on which all four vertices lie.

6. The perpendicular bisectors of any two chords of the arc will intersect at the center of the circle.

503 7.





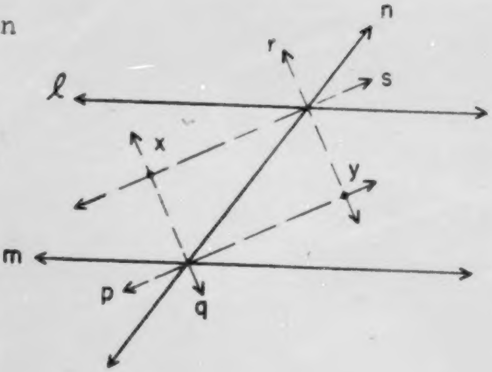
Let d be the length of the given segment. Using any square find d', the difference between the diagonal and side. In the proportion $\frac{d'}{d} = \frac{s'}{s}$, s will be the length of the side of the required square.

504 8. No, not if $a > b + AB$ or $b > a + AB$.

9. Consider a circle with center P and radius $\overline{PA}$. A, B, C and D will lie on this circle. Since parallel lines intercept congruent arcs, $m\widehat{AB} = m\widehat{CD}$ and $m\widehat{BC} = m\widehat{AD}$. Hence, $m\widehat{AB} + m\widehat{BC} = m\widehat{CD} + m\widehat{AD}$. Hence, $\widehat{AC}$ is a semi-circle and $m\angle B = 90$ so the parallelogram is a rectangle.

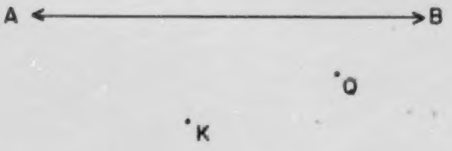
10. Consider a circle with center P and radius $\overline{PA}$. The parallel chords $\overline{AB}$ and $\overline{CD}$ intercept congruent arcs $\widehat{AD}$ and $\widehat{BC}$. These arcs have congruent chords so that the trapezoid is isosceles. Conversely, only one such point P exists for a given isosceles trapezoid.

504 11. Let ℓ, m, be the given parallel lines, and n the transversal. Any point equidistant from m and n must lie on one of the bisectors p, q, of the angles determined by m, n. Similarly, any point equidistant from ℓ and n must lie on one of the bisectors r, s of the angles determined by ℓ, n. Thus, any point equidistant from ℓ, m, n, must lie on the intersection of the set A, consisting of lines p and q, and set B, consisting of lines r and s. Since these lines are parallel in pairs (easily proved) the intersection of sets A and B consists of two points only. In the diagram these are the points X and Y where q intersects s and r intersects p.



Illustrative Test Items for Chapter 14

A. 1. Given $\overleftrightarrow{AB}$ and points K and Q in plane E. Tell how to locate a point on $\overleftrightarrow{AB}$ which is equidistant from K and Q.



2. Consider all circles in one plane tangent to $\overleftrightarrow{AB}$ at A. Describe the set of points which are centers of the circles.

3. Describe the set of centers of circles in one plane with radius 3 which are tangent to $\overleftrightarrow{AB}$.

4. Describe the set of points in the plane which are equidistant from the sides of $\angle ABC$ and at distance x from B.

5. If two parallel planes are d units apart, what will be the length of the radii of spheres tangent to both planes? Describe the set of centers of spheres tangent to both planes.

6. Describe the set of points which are at distance 5 from A and at distance 6 from B.

7. Given right ΔABC with $\overline{AB}$ as hypotenuse. Describe the set of points C in the plane of the triangle; in space.

8. Describe the set of mid-points of parallel chords in a circle.

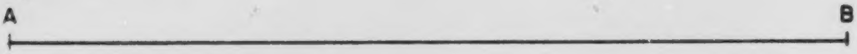
9. Under what conditions will the centers of circles inscribed in and circumscribed about a triangle be the same point?

10. Describe the set of centers of circles tangent to the sides of an angle.

11. Under what conditions will one vertex of a triangle be the intersection point of the altitudes of the triangle?

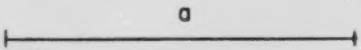
12. Under what conditions will the points of concurrency of altitudes, medians and angle bisectors of a triangle be the same point?

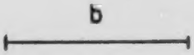
B. 1.



Construct an isosceles triangle in which the base is half the length of one of the congruent sides and for which AB is the length of the perimeter.

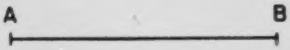
2.





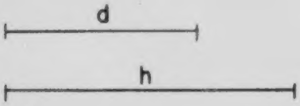
Construct a rhombus in which the lengths of the diagonals are a and b.

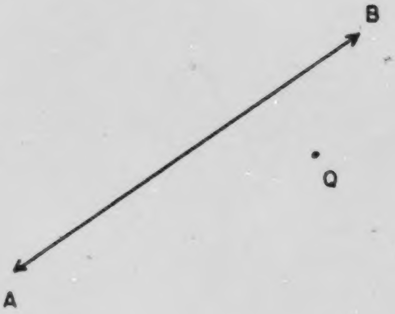
3. Construct an isosceles triangle with base $\overline{AB}$ and base angles each measuring 75.



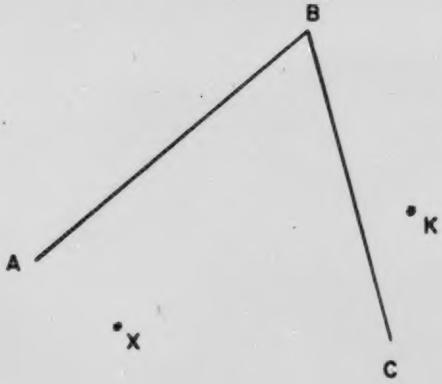
C. If problems are chosen from this section, we suggest giving each student a mimeographed sheet on which the problems are arranged and on which the student does the constructions. This will make the papers easier to check.

1. By construction locate points at distance d from $\overleftrightarrow{AB}$ and at distance h from Q.



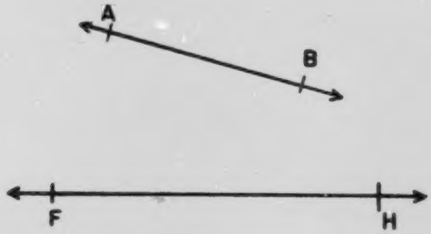


2. By construction locate points which are equidistant from $\overrightarrow{AB}$ and $\overrightarrow{BC}$ and equidistant from X and K, as shown.

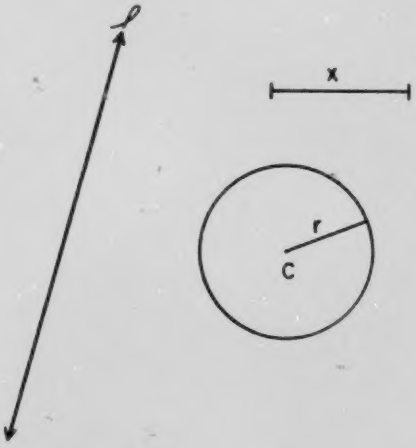


3. $\overleftrightarrow{AB}$ and $\overleftrightarrow{FH}$ intersect at some inaccessible point C.

By construction determine the bisector of $\angle$ ACF.



4. Given line ℓ and circle C, as shown. Construct a circle of radius x tangent to ℓ and C.



Answers

A\. 1. The intersection of $\overleftrightarrow{AB}$ and the perpendicular bisector of $\overline{KQ}$ is the point in question. If $\overline{KQ} \perp \overleftrightarrow{AB}$ there will either be no such or an infinite number.

2. The line perpendicular to $\overleftrightarrow{AB}$ at A except point A.

3. Two lines parallel to $\overleftrightarrow{AB}$ and at the distance 3 from $\overleftrightarrow{AB}$.

4. The intersection of the bisector of $\angle ABC$, and the circle with center B and radius x. There is one point.

5. $\frac{1}{2}$ d. The plane parallel to both given planes and midway between them.

6. The intersection of the sphere with center A and radius 5, and the sphere with center B and radius 6. If $AB < 11$, this intersection will be a circle. If $AB = 11$, the intersection will be one point. If $AB > 11$ there will be no intersection.

7. The circle whose diameter is $\overline{AB}$ minus A and B. The sphere whose diameter is $\overline{AB}$ minus A and B.

8. The diameter perpendicular to one of the chords, minus the end-points of the diameter.

9. If and only if the triangle is equilateral.

10. The bisector of the angle minus the vertex of the angle.

11. If and only if the triangle is a right triangle.

12. If and only if the triangle is equilateral.

B. 1. Divide $\overline{AB}$ into 5 congruent segments (Theorem 14-11). Use $\frac{1}{5}$ AB as base and then using $\frac{2}{5}$ AB as radius and A and B as centers construct intersecting arcs to locate a third vertex of the triangle.

2. Let AB = a. Let M be the mid-point of $\overline{AB}$. On the perpendicular bisector of $\overline{AB}$ make QM = XM = $\frac{1}{2}$ b. Then AXBQ is the required rhombus.

3. Construct an angle whose measure is 60. By bisecting get angles with measures 30 and 15, hence 75 = 60 + 15. At A and B construct angles with measure 75.

C. 1. Construct lines parallel to $\overleftrightarrow{AB}$ at distance d. Construct the circle Q with radius h. The points required are the intersections of the parallels and the circle.

2. One point, the intersection of the bisector of $\angle$ ABC and the perpendicular bisector of $\overline{XK}$.

3. Construct lines ℓ and ℓ_1 parallel to $\overleftrightarrow{AB}$ and $\overleftrightarrow{FH}$ at the same distance from $\overleftrightarrow{AB}$ and $\overleftrightarrow{FH}$. If ℓ and ℓ_1 intersect at Q, the bisector of $\angle$ Q will be the required bisector since each of its points is equidistant from $\overleftrightarrow{AB}$ and $\overleftrightarrow{FH}$.

4. Construct parallels to ℓ at distance x from it. With C as center construct the circle whose radius is r + x. The intersections of this circle and either parallel will be centers of circles of radius x tangent to ℓ and C.

Chapter 15

AREAS OF CIRCLES AND SECTORS

In this chapter we study the length and area of a circle, the length of a circular arc and the area of a circular sector, deriving the familiar formulas. The necessary treatment of limits is left at an intuitive level. We study the measurement of a circle in the familiar way by means of inscribed regular polygons and so the chapter begins by discussing the idea of polygon. This has not been needed earlier since the idea of polygonal region (Chapter 11) was sufficient for our purposes.

506 We want a polygon to be a simple "path" that doesn't cross itself. Property (1) takes care of this, since, it prevents two segments from crossing. Property (2) is included for simplicity of treatment. For example, suppose P_2, P_3, P_4 were permitted to be collinear. Then, in the face of Property (1), $\overline{P_2P_3}$ and $\overline{P_3P_4}$ would be collinear segments having only P_3 in common so that the union of $\overline{P_2P_3}$ and $\overline{P_3P_4}$ would simply be the segment $\overline{P_2P_4}$ and there would be no need to introduce P_3 in the definition at all.

As we indicated in Chapter 11, there is a close
connection between the ideas of polygon and polygonal region:
The union of any polygon and its interior is a polygonal
region. Although this seems quite obvious intuitively, it
is very difficult to prove since there is no simple way to
507 define interior of a polygon. However, for a convex polygon
it is relatively easy to define interior and to see what is
involved in a proof of the principle stated above. (See
508 Problem 3 of Problem Set 15-1.)

Problem Set 15-1

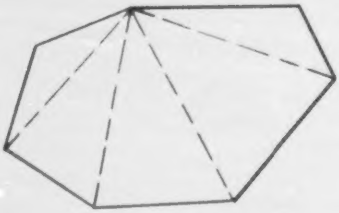
508 1. It has 6 sides, but only 5 vertices.

509 2. Yes. 12. 12. All sides have the same length. All angles are right angles.

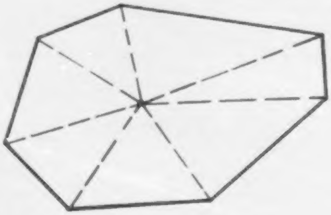
*3. a. By definition of a convex polygon, given any side of the polygon, the entire polygon, except for that one side, lies entirely in one of the half-planes determined by that side. The intersection of all such half-planes is the interior of the polygon.

Alternatively:
The intersection of the interiors of all the angles of the polygon is the interior of the polygon.

b.



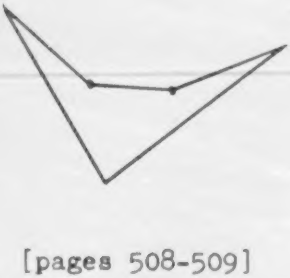
or



indicate ways in which any convex polygon and its interior can be cut into triangular regions.

4. a. 0, 2, 5, 9, 5150, $\frac{m(m - 3)}{2}$. (A diagonal of an n-gon can be drawn from each vertex to all but three other vertices. In doing this, each diagonal is counted twice.)

b.

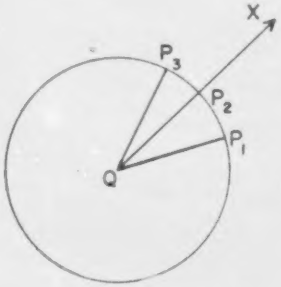


509 5. Since the polygon is convex its diagonals lie in the interior of each angle, so that the Angle Addition Postulate can be applied to show the sum of the angles of the polygon equals the sum of the angles of the triangles. Consider the point from which the diagonals are drawn, the vertex of each triangle and the opposite side the base. An n-gon then has n-2 such bases, and therefore there are n-2 triangles. Since the sum of the angles of each is 180, the sum of the angles of the polygon is $(n - 2) \cdot 180$.

6. The number of triangles formed with vertex Q is the same as the number of sides of the n-gon, so that the sum of the angles of the triangle is 180n. The sum of the angles at Q is 360. Hence, the sum of the angles of the polygon is $180n - 360 = 180(n - 2)$.

---

510 We indicate how a circle can be divided into n congruent arcs end to end. Let Q be the center of the circle and $\overline{QP_1}$ a given radius. Let H_1 be a half-plane lying in the plane of the circle with edge $\overleftrightarrow{QP_1}$. By the Angle Construction Postulate there is a point X in H_1 such that $m\angle P_1QX = \frac{360}{n}$.



By the Point Plotting Theorem, there is a point P_2 on $\overrightarrow{QX}$ such that $QP_2 = QP_1$. Then the minor arc $\overset{\frown}{P_1P_2}$ has measure $\frac{360}{n}$. Now repeat the process replacing P_1 by P_2 and half-plane H_1 by H_2, the half-plane opposite to P_1, with edge $\overleftrightarrow{QP_2}$. This yields a minor arc $\overset{\frown}{P_2P_3}$ of measure $\frac{360}{n}$ which intersects $\overset{\frown}{P_1P_2}$ only in P_2. Continuing in this way we get a sequence of points P_1, P_2, P_3, ..., $P_{n-1}P_n$, such that successive minor arcs $\overset{\frown}{P_1P_2}$, $\overset{\frown}{P_2P_3}$, ..., $\overset{\frown}{P_{n-1}P_n}$ have measure $\frac{360}{n}$ and have in common only an end-point. Then the major arc $\widehat{P_1P_n}$ has measure $\frac{n-1}{n}\cdot 360$ and the measure of the minor arc $\widehat{P_1P_n}$ must be $\frac{1}{n}\cdot 360$. Thus the points P_1, P_2, ..., P_{n-1}, P_n divide the circle into n congruent arcs, end to end.

511 An inscribed polygon whose sides are congruent and whose angles are congruent can be proved to be convex, and so is regular in accordance with our definition. We do not prove this because we do not need it for our application of regular polygons to circles.

512 We speak of the regular n-gon inscribed in a given circle. Obviously there are many such regular n-gons for a given n, but they all are congruent and have congruent sides, congruent angles, and equal apothems, perimeters and areas.

The apothem of a regular polygon can also be described as the distance from the center to a side, or the radius of the inscribed circle of the polygon.

We write "A subscript n" here to emphasize that the area of the regular n-gon depends on the value assigned to n and to distinguish it from the area of the circle (circular region) which is denoted by A (see Section 15-4). Of course a, the apothem of the regular n-gon, and p, its perimeter, also depend on n and could be written a_n and p_n.

Problem Set 15-2

512 1. $\frac{1}{8}$.

2. a. $\frac{1}{8} \cdot 360 = 45$.

   b. Draw a circle and construct eight 45° central angles. Join in order the points where the sides of the angles intersect the circle.

   c. Draw a circle and construct two perpendicular diameters. Bisect the four right angles formed. Join in order the points where the sides of the resulting angles intersect the circle.

3. Draw a circle and construct five 72° central angles. Join in order the points where the sides of the angles intersect the circle.

4. $\frac{(n - 2)180}{n}$.

5. No. It is a 12-sided polygon all of whose sides are congruent and all of whose angles are congruent, but it is not convex.

513 6.

3.	120.	30.	60.
4.	90.	45.	90.
5.	72.	54.	108.
6.	60.	60.	120.
8.	45.	$67\frac{1}{2}$.	135.
9.	40.	70.	140.
10.	36.	72.	144.
12.	30.	75.	150.
15.	24.	78.	156.
18.	20.	80.	160.
20.	18.	81.	162.
24.	15.	$82\frac{1}{2}$.	165.

514 7. a. 6.

b. Regular hexagonal regions. 3.

c. Two pentagons and a decagon.
Two 12-gons and an equilateral triangle.
Two octagons and a square.

d. Three polygons with different numbers of sides may be used: 4, 6, 12; 4, 5, 20; 3, 7, 42; 3, 8, 24; 3, 9, 27; 3, 10, 15.

8. The measure of each exterior angle is 180 less the measure of an interior angle.
Adding n of these we get

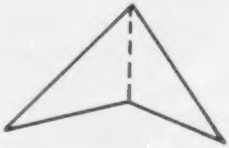
$$n \cdot 180 - \text{sum of the measures of the interior angles},$$

or $n \cdot 180 - (n - 2)\,180 = 360$.

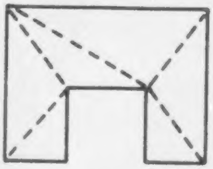
*9. a. $\frac{n}{2} - 1$ or $\frac{n - 2}{2}$.

b. $(\frac{n}{2} - 1) \cdot 360 = (n - 2)180$.

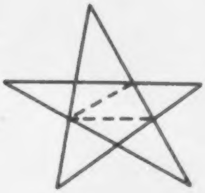
515 10. a. $n = 4$,
$S = 2 \cdot 180 = (n - 2)180$.



b. $n = 8$,
$S = 6 \cdot 180 = (n - 2)180$.



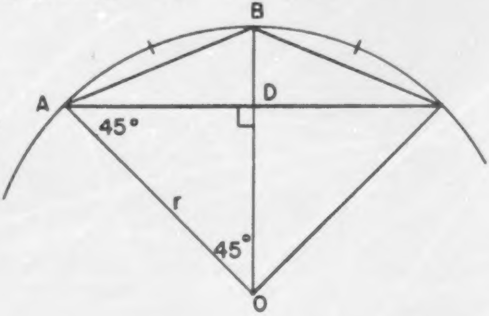
c. $n = 10$,
$S = 8 \cdot 180 = (n - 2)180$.



515 11. The angle sum is increased by 180 while the number of sides is increased by one.

12. The radius is also 2. The apothem is the altitude of an equilateral triangle with side 2, or $\sqrt{3}$.

*13. In the figure, side $\overline{AB}$ of a regular inscribed octagon is 1 unit long. Since Δ ADO is a right isosceles triangle, $AD = DO = \frac{r}{\sqrt{2}}$. $BD = r - \frac{r}{\sqrt{2}}$. In right triangle ABD, $AD^2 + BD^2 = AB^2$ or $\left(\frac{r}{\sqrt{2}}\right)^2 + \left(r - \frac{r}{\sqrt{2}}\right)^2 = 1$, from which $r = \sqrt{\frac{1}{2 - \sqrt{2}}}$, or approximately 1.3.



Beginning in Section 15-3 the text introduces the notion of a limit. It is not intended that the students be given a formal treatment of limits, but rather that they develop an intuitive idea of what a limit is. A discussion like the following may be helpful.

516 When we write $p \longrightarrow C$, we have in mind that C is a fixed number, the length (or circumference) of the circle, but that there are many successive values for p, depending on which inscribed regular n-gon we are considering. So it is desirable to write p_n instead of p for the perimeter of the inscribed regular n-gon. Then we say $p_n \longrightarrow C$, meaning that the successive numbers p_n approach C as a limit. Observe that we have an infinite sequence or progression of numbers which are the perimeters of regular inscribed polygons for successive values of n; we begin with n = 3, giving us an inscribed equilateral triangle,

then $n = 4$ yields an inscribed square and so on. We represent the infinite sequence p_n as p_3, p_4, ..., p_n, ... and we think of these numbers as being approximations to C which get better and better as we run down the sequence. As a simple analog consider the infinite sequence

$$.3, .33, .333, .3333, .33333, \ldots$$

which arises when we divide 1 by 3 and take the successive decimal quotients. These numbers are approximations to $\frac{1}{3}$ which get better and better as we travel down the sequence and we may say that this sequence approaches $\frac{1}{3}$ as a limit. Other examples are the two sequences

$$1, \frac{1}{2}, \frac{1}{4}, \ldots , \frac{1}{2^n}, \ldots$$

$$1, 1\tfrac{1}{2}, 1\tfrac{3}{4}, \ldots$$

which have limits 0 and 2. The essential point in all four cases is that each sequence has a uniquely determined "boundary" or "limit number" and that we can reason about the limit of a sequence if we know the sequence, that is, if its successive numbers are determined. However, we can not assume that every sequence has a limit. For example, the following sequence has no limit: 1, -2, 4, -8, 16, ...

We need three basic properties of sequences:

(I) If a sequence has a limit it has a unique limit.

(II) If sequence $a_n \longrightarrow a$, then sequence $Ka_n \longrightarrow Ka$ for any fixed number K.

(III) If sequence $a_n \longrightarrow a$ and sequence $b_n \longrightarrow b$ then sequence $a_n b_n \longrightarrow ab$.

Property (I) says in effect that if the terms of a sequence are getting closer and closer to a number a, they can't, at the same time, be getting closer and closer to another number b. As an illustration of (II) observe that

$$.3, .33, .333, \ldots \longrightarrow \frac{1}{3}$$

and that the sequence of "doubles" has double the limit:

$$.6, .66, .666, \ldots \longrightarrow \frac{2}{3}.$$

To illustrate (III) consider

$$6, 5.1, 5.01, 5.001, \ldots \longrightarrow 5,$$

$$4,\ 3.1,\ 3.01,\ 3.001,\ \ldots \longrightarrow 3.$$

You will easily convince yourself that the sequence of products of corresponding terms approaches $15 = 5 \cdot 3$.

Notice that in the discussions concerning limits, no mention of "infinity" is made.

The concept of a limit does not involve any notion of infinity. While the word and the symbol (∞) for it are convenient in certain branches of higher mathematics, they should be avoided in introductory discussions where they are neither useful nor enlightening.

517 The properties of limits used here are easily clarified. Let us write p_n for p and p_n' for p' to emphasize that we have two sequences of perimeters, one for each circle. Further, we have $p_n \longrightarrow C$ and $p_n' \longrightarrow C'$, and

$$\frac{p_n}{r} = \frac{p_n'}{r'}.$$

Now we apply Property (II) above to $p_n \longrightarrow C$ taking $K = \frac{1}{r}$ and get $\frac{p_n}{r} \longrightarrow \frac{C}{r}$. Similarly, $p_n' \longrightarrow C'$ yields $\frac{p_n'}{r'} \longrightarrow \frac{C'}{r'}$. To summarize, we have sequences

$$\frac{p_1}{r},\ \frac{p_2}{r},\ \ldots,\ \frac{p_n}{r},\ \ldots \longrightarrow \frac{C}{r}$$

$$\frac{p_1'}{r'},\ \frac{p_2'}{r'},\ \ldots,\ \frac{p_n'}{r'},\ \ldots \longrightarrow \frac{C'}{r'},$$

whose corresponding terms are the same numbers. That is, the sequences are the same. Thus, by Property (I) they must have the same limit. Therefore

$$\frac{C}{r} = \frac{C'}{r'}.$$

518 For a treatment of irrational numbers, see the forthcoming book, Irrational Numbers, by Ivan Niven to be published by Random House and the Wesleyan University Press.

Problem Set 15-3

518 1. a. The radius of the circle.

b. 0.

c. 180.

d. The circumference of the circle.

2. $C = 2\pi r,$

$628 = 6.28r,$

$100 = r.$

The radius of the pond is approximately 100 yards.

519 3. $\frac{22}{7}$ is the closer approximation.

$\frac{22}{7} = 3.1429-,$

$\pi = 3.1416-,$

$3.14 = 3.1400.$

4. $C = 2\pi r = 480{,}000\pi$. The circumference is approximately 1,500,000 miles.

5. The formula gives $2\pi r = 6.28 \times 93 \cdot 10^6 = 584 \cdot 10^6$ or 584 million miles, approximately.

Our speed is about 67,000 miles per hour.

6. The radius of the inscribed circle is 6 so that its circumference is 12π. The radius of the circumscribed circle is $6\sqrt{2}$ so that its circumference is $12\pi\sqrt{2}$.

*7. The perimeter of PQRS is greater than the circumference of the circle.

$AD = 2$ and $XW = \sqrt{2}$. Hence $PS = \frac{1}{2}(2 + \sqrt{2})$.

The perimeter of the square is $2(2 + \sqrt{2})$.

The circumference of the circle is 2π. But $2 + \sqrt{2} > \pi$.

8. The increase in circumference is 2π in each case.

[pages 518-519]

520 Justification of limit properties used in Theorem 15-2: We have, writing a_n for a and p_n for p, $a_n \longrightarrow r$ and $p_n \longrightarrow C$. By Property III (see above) $a_n p_n \longrightarrow rC$, and by Property II, $\frac{1}{2} a_n p_n \longrightarrow \frac{1}{2} rC$. Since $A_n = \frac{1}{2} a_n p_n$, by substitution we get

$$A_n \longrightarrow \frac{1}{2} rC.$$

But we have $A_n \longrightarrow A$. Since by Property I sequence A_n can have only one limit, $A = \frac{1}{2} rC$.

Problem Set 15-4

522 1. a. $C = 10\pi$, $A = 25\pi$.

b. $C = 20\pi$, $A = 100\pi$.

2. a. $C = 2\pi n$, $A = \pi n^2$.

b. $C = 20\pi n$, $A = 100\pi n^2$.

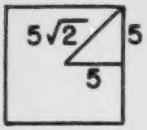
3. a. $4\pi - \pi = 3\pi$. The area would be approximately 9.4 square cm.

b. No.

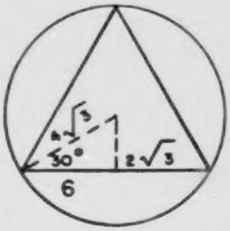
4. The area of the first is 9 times the area of the second.

5.
$$C = 2\pi r = 20.$$
$$r = \frac{10}{\pi}.$$
$$\text{Area of circle} = \frac{100}{\pi}$$
$$= 32 \text{ approx.}$$
$$P = 4s = 20$$
$$s = 5.$$
Area of square = 25.

The area of the circle is greater by about 7 square inches.



6. $\pi(5\sqrt{2})^2 - \pi(5)^2 = 25\pi$.
The area is 25π square inches.



523 7. Radius = $4\sqrt{3}$ inches.
Circumference = $8\sqrt{3}\pi$ inches.
Area = 48π square inches.

8. It is only necessary to find the square of the radius of the circle. If a radius is drawn to a vertex of the cross it is seen to be the hypotenuse of a right triangle of sides 2 and 6. The square of the radius is therefore $2^2 + 6^2 = 40$. The area of the circle is therefore 40π, 125.6 approximately. The required area is therefore $125.6 - 80 = 40.6$.

9. Draw $\overline{PB}$ and $\overline{PC}$. The area of the annulus is $\pi(PC)^2 - \pi(PB)^2$, the difference of the areas of the two circles. This can also be written $\pi(PC^2 - PB^2)$. By Pythagorean Theorem, $PC^2 - PB^2 = BC^2$. Therefore the area of the annulus is πBC^2.

10. The section nearer the center of the sphere will be the larger.

$$r^2 = (10)^2 - (5)^2.$$

$$r_1{}^2 = (10)^2 - (3)^2.$$

Therefore, $r_1 > r$.

11. $\frac{s^2}{2}$.

524 *12. $AC^2 + BC^2 = AB^2$.

$$\frac{\pi}{8} AC^2 + \frac{\pi}{8} BC^2 = \frac{\pi}{8} AB^2.$$

$$(r + g) + (h + s) = g + h + t.$$

$$r + s = t.$$

524 *13. a. Note that $r_1 = OA = OR = BP$ and $r_2 = OS = CP$. By successive use of the Pythagorean Theorem we get $r_1 = r\sqrt{2}$, $r_2 = r\sqrt{3}$, $r_3 = r\sqrt{4}$.

b. Now, using the area formula for a circle, we have

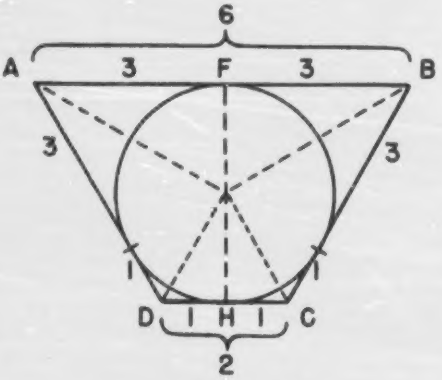
$a = \pi r^2$;

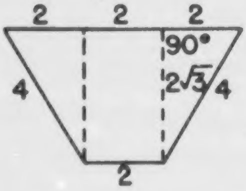
$b = \pi(r\sqrt{2})^2 - a = \pi r^2$;

$c = \pi(r\sqrt{3})^2 - (a + b) = 3\pi r^2 - 2\pi r^2 = \pi r^2$;

$d = \pi(2r)^2 - (a + b + c) = 4\pi r^2 - 3\pi r^2 = \pi r^2$.

14. From the second figure, $(4)^2 - (2)^2 = 12$, and so the altitude of the trapezoid is $2\sqrt{3}$. In the first figure, since the bases are parallel and tangent to the circle we see that $\overline{FH}$ (altitude of the trapezoid) must be a diameter, and so the radius is $\sqrt{3}$. Area of the circle is, then, 3π. Area of the trapezoid is $8\sqrt{3}$. The area outside the circle is $(8\sqrt{3} - 3\pi)$ square inches. This is approximately 4 square inches.





525 Notice the common procedure in treating length of circle and length of arc. In each case we "approximate" by means of chords of the same length.

526 The agreement to consider a circle as an "arc", enables us to include in Theorem 15-3 the case of the whole circle as an arc of measure 360.

To illustrate the application of Theorems 15-3 and 15-4 assign Problems 1, 3, 6 and 7.

527 One concrete illustration of a sector of a circle is a lady's fan, with the ribs of the fan standing for the segments $\overline{QP}$. The arc $\widehat{AB}$, of course, need not be a minor arc. Observe that the definition can also be phrased: If $\widehat{AB}$ is an arc of a circle with center Q then the set of all points X each of which lies in a segment joining Q to a point of $\widehat{AB}$ is a sector.

Problem Set 15-5

527 1. 5π, 7.5π, 6π, 3π.

2. 9π, $.1\pi$.

3. $\frac{3}{\pi}$ in each case.

528 4. The measure of the arc is 90. The length of the arc is π.

5. a. Area of sector $= \frac{1}{6}\pi \cdot 12^2 = 24\pi$.

Area of triangle $= \frac{12^2}{4}\sqrt{3} = 36\sqrt{3}$.

Area of segment $= 24\pi - 36\sqrt{3}$ or 13.04.

b. Area of sector $= \frac{1}{3}\pi \cdot 6^2 = 12\pi$.

Area of triangle $= \frac{1}{2} \cdot 6\sqrt{3} \cdot 3 = 9\sqrt{3}$.

Area of segment $= 12\pi - 9\sqrt{3}$ or 22.11.

527 c. Area of sector $= \frac{1}{8}\pi \cdot 8^2 = 8\pi$.

Area of triangle $= \frac{1}{2}8 \cdot 4\sqrt{2} = 16\sqrt{2}$.

Area of segment $= 8\pi - 16\sqrt{2}$ or 2.51.

6. a. 2π. b. π.

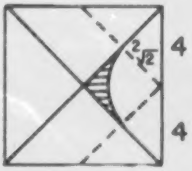
529 7. Draw $\overline{BG} \perp \overline{AC}$. Then $GC = 6$, $AG = 24$. In the right triangle ΔAGB, the length of the hypotenuse is twice the length of one leg, so $m\angle ABG = 30$, $m\angle BAG = 60$, and $CE = GB = 24\sqrt{3}$. The major arc $\overset{\frown}{CD}$ has the length $\frac{2}{3}(2\pi \cdot 30) = 40\pi$ and the minor arc $\overset{\frown}{EF}$ has the length $\frac{1}{3}(2\pi \cdot 6) = 4\pi$. Thus, the total length of the belt is

$$2(24\sqrt{3}) + 40\pi + 4\pi = 48\sqrt{3} + 44\pi.$$

The belt is approximately 221 inches long.

8. To find one small shaded area subtract the area of a 90° sector whose radius is $2\sqrt{2}$ from the area of a square whose side is $2\sqrt{2}$.

$$(2\sqrt{2})^2 - \frac{\pi(2\sqrt{2})^2}{4} = 8 - 2\pi.$$



The area of the shaded area is $4(8 - 2\pi)$. This is approximately 6.87 square inches.

Review Problems

530 1. The first and third are polygons. The third is a convex polygon.

2. a. Yes. c. No.

b. Yes.

3. 108, 120, 135, 144.

4. 12.

530 5. a. The regular octagon in each case.

b. The apothems are equal. The square has the greater perimeter.

6. From the formula $A = \frac{1}{2}ap$ for the area of a regular polygon.

7. 2π.

531 8. 1 and 2.

9. a. 72. b. $\frac{360}{n}$.

10. 2.

11. a. 10 to 1. c. 100 to 1.

b. 10 to 1.

12. 5, $\frac{5\pi}{3}$.

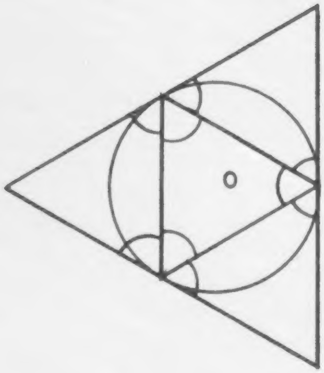
13. $A = \pi r^2$. $r = \frac{1}{2}d$.

Hence, $A = \pi(\frac{1}{2}d)^2 = \frac{1}{4}\pi d^2$.

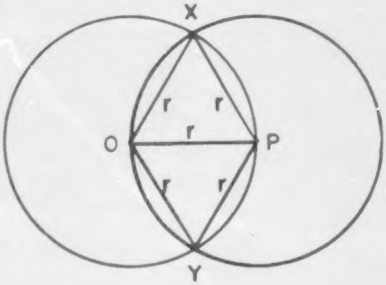
14. 15π inches, a distance equal to $\frac{3}{4}$ of its circumference.

15. 4π and $\frac{2}{3}\pi$.

16. There are several methods of showing that the four small triangles are congruent to each other. For example, each of the angles marked with an arc will have a measure of 60. In this case the congruence is by A.S.A. Hence, each of the four small triangles has the same area, and then the circumscribed triangle has an area four times that of the inscribed triangle.



531 *17. The woodchuck's burrow will be in the region bounded by $\widehat{XOY}$ and $\widehat{XPY}$.



The area of each of the equilateral triangles is $\frac{r^2}{4}\sqrt{3}$. The area of each segment is $\frac{1}{6}\pi r^2 - \frac{r^2}{4}\sqrt{3}$. Then the area in which the woodchuck can settle is $2(\frac{r^2}{4}\cdot\sqrt{3}) + 4(\frac{1}{6}\pi r^2 - \frac{r^2}{4}\sqrt{3}) = (\frac{2}{3}\pi - \frac{1}{2}\sqrt{3})r^2$, as any woodchuck knows.

18. Let a and p be the apothem and perimeter of the smaller polygon and a' and p' be the apothem and perimeter of the larger polygon. The ratio of the areas is $\frac{ap}{a'p'}$. But $\frac{a}{a'} = \frac{p}{p'}$, so that, the ratio of the areas is $\frac{p^2}{p'^2}$. Hence, $\frac{p}{p'} = \sqrt{\frac{8}{18}} = \sqrt{\frac{4}{9}} = \frac{2}{3}$. The sides also have the ratio $\frac{2}{3}$.

Illustrative Test Items for Chapter 15

A. Indicate whether each of the following is true or false.

1. The ratio of circumference to radius is the same number for all circles.
2. If the number of sides of a regular polygon inscribed in a given circle is increased indefinitely, its apothem approaches the radius of the circle as a limit.
3. Any polygon inscribed in a circle is a regular polygon.
4. A polygon is a polygonal region.

5. If the radius of one circle is three times that of a second, then the circumference of the first is three times that of the second.

6. The area of a square inscribed in a given circle is half the area of one circumscribed about the circle.

7. In the same circle, the areas of two sectors are proportional to the squares of the measures of their arcs.

8. The ratio of the area of a circle to the square of its radius is π.

9. The length of an arc of a circle can be obtained by dividing its angle measure by π.

10. Doubling the radius of a circle doubles its area.

B. 1. Find the measure of an angle of a regular nine-sided polygon.

2. Into how many triangular regions would a convex polygonal region with 100 sides be separated by drawing all possible diagonals from a single vertex?

3. If the circumference of a circle is a number between 16 and 24 and the radius is an integer, find the radius.

4. If the number of sides of a regular polygon inscribed in a circle is increased without limit, what is the limit of the length of one side? of its perimeter?

5. Write a formula for the area of a circle in terms of its circumference.

6. If the area of a circle is 2π, find its radius.

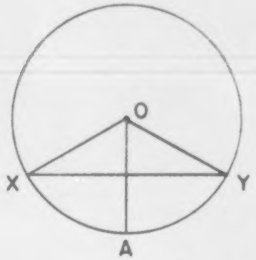
7. The area of one circle is 100 times the area of a second. What is the ratio of the diameter of the first to that of the second?

8. The angle of one sector of a circle is 50°. The angle of a second sector of the same circle is 100°. Find the ratio of the length of the arc of the first sector to that of the second, and the ratio of the area of the first sector to that of the second.

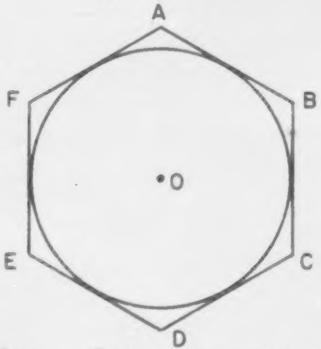
9. A circular lake is 2 miles in diameter. If you walk at 3 miles per hour, about how many hours will it take to walk around it? (Give the answer to the nearest whole number.)

10. An angle is inscribed in a semi-circle of radius 6. What is the least possible value of the sum of the areas of the two circular segments that are formed?

C. 1. In circle O, chord $\overline{XY}$ is the perpendicular bisector of radius $\overline{OA}$. $OA = 6$. Find $m\widehat{XAY}$, the length of $\widehat{XAY}$, the area of sector XOY, and the area of the region bounded by $\overline{XY}$ and $\widehat{XAY}$.



2. ABCDEF is a regular hexagon circumscribed about circle O. If its perimeter is 12, find the circumference and the area of the circle.

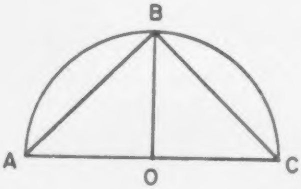


3. On an aerial photograph the surface of a reservoir is a circle with diameter $\frac{7}{8}$ inch. If the scale of the photograph is 2 miles to 1 inch, find the area of the surface of the reservoir. (Use $\frac{22}{7}$ for π. Give the result to the nearest one-half square mile.)

Answers

A. 1. True.
2. True.
3. False.
4. False.
5. True.
6. True.
7. False.
8. True.
9. False.
10. False.

B. 1. 140.
2. 98.
3. 3.
4. 0. The circumference of the circle.
5. Since $C = 2\pi r$, $r = \frac{C}{2\pi}$.
Since $A = \pi r^2$, $A = \pi\left(\frac{C}{2\pi}\right)^2 = \frac{C^2}{4\pi}$.
6. $\sqrt{2}$.
7. 10 to 1.
8. 1 to 2 in each case.
9. 2.
10. The sum of the areas of the segments will be least when the area of $\triangle ABC$ is greatest. In this case the altitude to $\overline{AC}$ is the radius of the circle. The sum of the areas of the segments is found by subtracting the area of the triangle from that of the semicircle. The result is $18\pi - 36$.



C. 1. $m\widehat{XAY} = 120$. The length of $\widehat{XAY} = 4\pi$.
Area sector XOY $= 12\pi$. Area segment XAY $= 12\pi - 9\sqrt{3}$.

2. The radius of the circle is the altitude of equilateral triangle $\triangle$ OAB, so that, $r = \sqrt{3}$. Hence $C = 2\pi\sqrt{3}$ and $A = 3\pi$.

3. The diameter of the reservoir in miles is $\frac{7}{8} \cdot 2 = \frac{7}{4}$, so that its radius is $\frac{7}{8}$. The area is
$\frac{22}{7} \cdot \frac{7}{8} \cdot \frac{7}{8} = \frac{77}{32}$. The area of the reservoir is about $2\frac{1}{2}$ square miles.

Chapter 16

VOLUMES OF SOLIDS

In this chapter we study mensuration properties of familiar solids: prisms, pyramids, cylinders, cones and the sphere. Our proofs are conventional in spirit, although our derivation of the formula for surface area of a sphere, based on an assumed approximation to the volume of a spherical shell, is quite unusual in an elementary text. We assume Cavalieri's Principle (Postulate 22) in order to avoid coming to grips with fundamental difficulties of a type occurring in Integral Calculus. We emphasize strongly analogies between prisms and cylinders, between pyramids and cones. In fact our definitions of prism and pyramid are formulated so as to be applicable to cylinder and cone. These figures are defined, quite precisely, as solids (spatial regions) rather than surfaces, since our basic concern is for volumes of solids rather than for areas of surfaces.

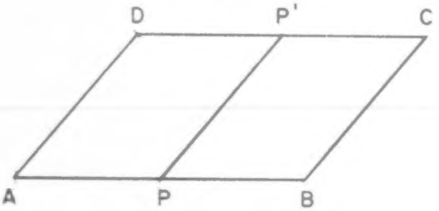
534 Notice that we define a prism directly as a solid (region of space) rather than as a surface (prismatic surface). This is quite natural since our main object of study in this chapter is volumes of regions, rather than areas of surfaces. This is analogous to our earlier emphasis on polygonal regions rather than polygons. Note how simply our definition generates the whole solid from the base polygonal region K, and how easily it enables us to pick out the "bounding surface", (see the definitions of lateral surface and total surface in the text). If we used the alternative approach and defined a prism as a surface we still would have the problem of defining the interior of this surface in order to get the corresponding solid. Similar observations hold for our treatment of pyramids, cylinders and cones.

535 Note that in our use of the word "cross-section", the intersecting plane must be parallel to the base. It is possible to have sections formed by a plane which is not parallel to the base, but such sections would not possess all the properties of a cross-section. Note that since a prism in our treatment is a solid, its cross-section is a polygonal region, not a polygon.

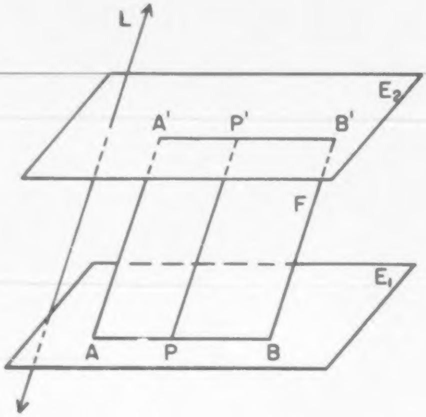
535 In Theorem 16-1, the text states that the cross-sections of a triangular prism are congruent to the base. Up to this point no mention has been made of congruence of triangular regions, but only of congruence of triangles. It is intuitively apparent that if two triangles are congruent, then their associated triangular regions also are congruent. This can be proved formally using the ideas of Appendix VIII. We will not speak of the congruence of polygonal regions other than triangular regions, since any polygonal region can always be divided into triangular regions.

536 Corollary 16-1-1 is a direct consequence of Theorem 16-1, since the upper base is a cross-section of the prism. A similar observation applies to Corollary 16-2-1.

537 A "parallelogram region" is defined formally as the union of a parallelogram and its interior. The interior of parallelogram ABCD consists of all points X which are on the same side of $\overleftrightarrow{AB}$ as C and D, on the same side of $\overleftrightarrow{BC}$ as D and A, on the same side of $\overleftrightarrow{CD}$ as A and B and on the same side of $\overleftrightarrow{DA}$ as B and C. An alternative definition which is suggested by the text definition of prism is the following: Let ABCD be a parallelogram. Then the union of all segments $\overline{PP'}$ where P is in $\overline{AB}$, P' is in $\overline{CD}$ and $\overline{PP'} \parallel \overline{AD}$ or $\overline{PP'} \parallel \overline{BC}$ is a parallelogram region.



537 Theorem 16-3 is easy to grasp intuitively, but tedious to prove formally. Here is an outline of a proof. Let E_1 and E_2 be the planes of the bases, L be the transversal and $\overline{AB}$ a side of the base. We want to show that the lateral face F which is the union of all segments $\overline{PP'}$, where P is in $\overline{AB}$, is a parallelogram region. Remember that by definition of a prism, $\overline{PP'} \parallel L$ and P' is in E_2. Consider $\overline{AA'}$ and $\overline{BB'}$ where $\overline{AA'} \parallel L$, $\overline{BB'} \parallel L$ and A', B' are in E_2. Then ABB'A' is a parallelogram and the lateral face F is the corresponding parallelogram region. To prove this, first show that every point P' is on $\overline{A'B'}$, and in fact that $\overline{A'B'}$ is the set of all such points P'. Then show that every point of $\overline{PP'}$ is on ABB'A' or is in its interior. Finally show that every point on ABB'A' or in its interior lies in some segment $\overline{PP'}$. Thus, the segments $\overline{PP'}$ constitute the parallelogram region composed of ABB'A' and its interior.



Problem Set 16-1

538 1. $\overline{FH} \parallel \overline{BA}$ (Definition of prism). Hence, $\overline{FH}$ and $\overline{BA}$ determine a plane (Theorem 9-1). By definition the upper and lower bases of a prism are parallel, hence, $\overline{FB} \parallel \overline{HA}$ (Theorem 10-1). Hence, $ABFH$ is a parallelogram.

539 2. $30 + 40 + 50 + 70 + 20 = 210$.

3. $3 \times 8 \times 10 + 8 \times 4\sqrt{3} = 240 + 32\sqrt{3}$.
The total surface area is $240 + 32\sqrt{3}$ square inches.

4. Since each lateral face is a rectangle, its area is the product of base and altitude. If e is the length of a lateral edge and S_1, S_2, S_3, ... are lengths of the sides of the prism base, then $A_1 = S_1e$, $A_2 = S_2e$, $A_3 = S_3e$, ... Adding these areas to get the lateral area, $A = S_1e + S_2e + \ldots = (S_1 + S_2 + S_3 + \ldots)e$. But $S_1 + S_2 + S_3 + \ldots = p$, the perimeter of the base. Therefore, $A = p \cdot e$.

5. 3, 6, $3\sqrt{3}$; 30, 90, 60; $\frac{9}{2}\sqrt{3}$.

6. Let the required perimeter be y inches. Since $52 = 10y$, we have $y = 5.2$. The perimeter of the base is 5.2 inches.

540 Cross-section is defined for pyramid exactly as for prism.

When we say in Theorem 16-4 that two triangular regions are similar, we mean of course that they are determined by similar triangles.

In (1) of Theorem 16-4, to justify $\overline{AP} \parallel \overline{A'P'}$ note that $E \parallel E'$ and that plane VAP intersects E and E' in $\overleftrightarrow{AP}$ and $\overleftrightarrow{A'P'}$. Thus, $\overleftrightarrow{AP} \parallel \overleftrightarrow{A'P'}$ by Theorem 10-1. Similarly in (2) we show $\overline{A'B'} \parallel \overline{AB}$.

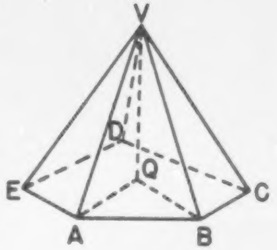
542 Our procedure in Theorem 16-5 is simply to split the pyramid into triangular pyramids and apply Theorem 16-4 to each of these.

Problem Set 16-2

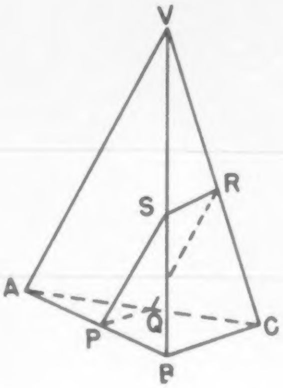
544 1. square; an equilateral triangle; 3.

2. 25 square inches.

3. $QA = QB$, $m\angle VQA = m\angle VQB = 90$; $\triangle VQA \cong \triangle VQB$ by S.A.S. Hence, $VA = VB$. Similarly, $VB = VC = VD = \ldots$, $AB = BC = CD = \ldots$ by definition, so $\triangle AVB \cong \triangle DVC \cong \triangle CVD \cong \ldots$ by S.S.S.



*4. Let P, Q, R and S be the mid-points of $\overline{AB}$, $\overline{AC}$, $\overline{VB}$ and $\overline{VC}$ respectively. Then $\overline{SR}$ and $\overline{PQ}$ are each parallel to $\overline{BC}$ and equal in length to $\frac{1}{2}BC$. Therefore, $\overline{SR}$ and $\overline{PQ}$ are parallel, coplanar, and equal in length making PQRS a parallelogram.



544 5. Let each edge of the base have length s. Each face is a triangle with base s and altitude a.

Hence, $A = \frac{1}{2}sa + \frac{1}{2}sa + \ldots$ or

$$A = \frac{1}{2}a(s + s + \ldots) = \frac{1}{2}ap.$$

545 6. By Theorem 16-5,

$$\frac{x}{336} = \frac{16}{49},$$

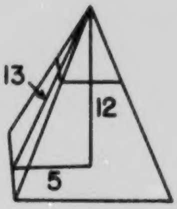
$$x = \frac{16 \cdot 336}{49} = \frac{768}{7} = 109\frac{5}{7}.$$

Area FGHJK = $109\frac{5}{7}$ square inches.

7. The altitude of each face is 13 inches by the Theorem of Pythagoras. Hence,

$4(\frac{1}{2} \cdot 10 \cdot 13) = 260.$

The lateral area is 260 square inches.



If x is the area of the cross-section 3 inches from the base then $\frac{x}{100} = (\frac{9}{12})^2 = \frac{9}{16}$ and $x = 56.25$.

Hence, its area is 56.25 square inches.

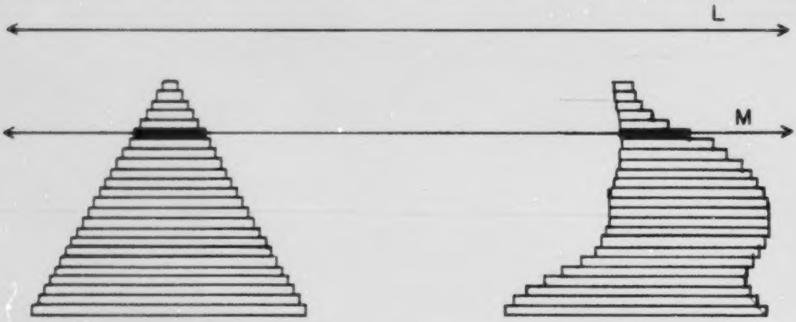
*8. Let $PK = a$ and $PB = b$. Draw altitude $\overline{PS}$. $\overline{PS} \perp JKLMN$ at R. $\overline{PB}$ and $\overline{PS}$ determine a plane which intersects JKLMN and ABCDE in $\overleftrightarrow{KR}$ and $\overleftrightarrow{BS}$ respectively. Since JKLMN $\parallel$ ABCDE, $\overleftrightarrow{KR} \parallel \overleftrightarrow{BS}$. In Δ PBS, by the Basic Proportionality Theorem, $\frac{PK}{PB} = \frac{PR}{PS}$. By Theorem 16-5, $\frac{\text{area JKLMN}}{\text{area ABCDE}} = (\frac{PR}{PS})^2$.

Hence, $\frac{\text{area JKLMN}}{\text{area ABCDE}} = (\frac{PK}{PB})^2 = (\frac{a}{b})^2$.

546 The text postulates the formulas for the volume of a rectangular parallelepiped and proceeds to prove the remaining formulas for the volumes of prisms, pyramids, cones, cylinders and spheres. This is analogous to the procedure followed in Chapter 11 when the formula for the area of a rectangle was postulated.

546-547 Cavalieri's Principle is an extremely powerful postulate. It can be proved as a theorem by methods resting on the theory of limits as developed in integral calculus. It will be used throughout the chapter to prove theorems concerning the volumes of solids.

A model for making Cavalieri's Principle seem reasonable can be made using thin rectangular rods in an approach slightly different from that of the text. Consider the following statement: Given a plane containing two regions and a line. If for every line which intersects the regions and is parallel to the given line the two intersections have equal lengths, then the two regions have the same area.



Here too, it should be pointed out that the approximations to the areas of the regions improve as the thickness of the rectangular rods becomes smaller and smaller. (Also, see Problem 8 of Problem Set 16-3.)

549 You may wish to point out that while the proofs of Theorems 16-7 and 16-8 require the solids to have their bases coplanar, in numerical application this is not necessary.

550 In the proof of Theorem 16-9, to help the students visualize how three triangular pyramids are formed by cutting a triangular prism, some visual aid should be used. Disected solids can be purchased from an equipment supply company; or one could try to make them by cutting up a bar of laundry soap. The three pyramids are formed by cutting the triangular prism by the planes through the points S, P, R and the points S, P, U.

551 Theorem 16-10 can be proved without recourse to Cavalieri's Principle by splitting the pyramid into triangular pyramids and applying Theorem 16-9. The proof in the text was chosen because it applies just as well to cones as to pyramids, (see Theorem 16-15).

Problem Set 16-3

552 1. $5 \times 4 \times \frac{3}{4} = 15$. 15 cubic feet of water in the tank.

$\frac{15 \times 1728}{231} = 112$ approx. 112 gallons approximately.

2. $20 \times 8 \times 4.6 = 736$. The volume is 736 cubic inches.

3. $\frac{2 \times 3 \times 3 \times 12 \times 12 \times 12}{2 \times 2 \times 231} = \frac{2592}{77} = 33.6$.

33 fish can be kept in the tank.

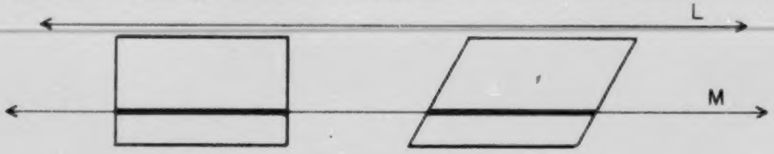
552 4. The base can be divided into six equilateral triangles with side 12. Therefore, altitude $\overline{QF}$ of Δ ABQ has length $6\sqrt{3}$. Since $QC = 9$, by Pythagorean Theorem $CF = \sqrt{189}$. Hence the lateral area is $\frac{1}{2} \cdot 72 \cdot \sqrt{189} = 36\sqrt{189}$. Now, $V = \frac{1}{3}Ah$, or $V = \frac{1}{3}(6 \cdot \frac{1}{2} \cdot 12 \cdot 6\sqrt{3}) \cdot 9 = 648\sqrt{3}$.

5. $1836 = \frac{1}{3} \cdot (18)^2 \cdot h$. or $h = 17$. The height is 17 feet.

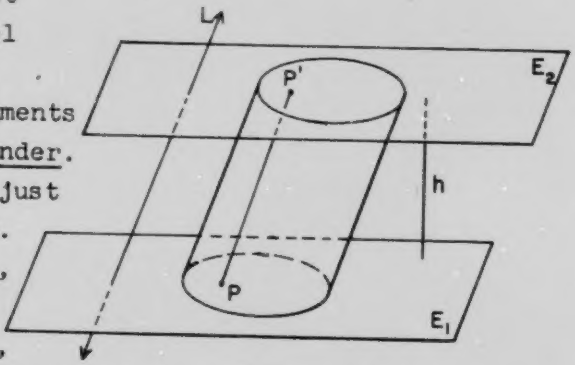
6. The lateral edges will also be bisected and therefore corresponding sides of the section and base will be in the ratio $\frac{1}{2}$, and the areas of the section and base in the ratio $\frac{1}{4}$. The volume of the pyramid above the section will be $\frac{1}{8}$ of that of the entire pyramid because its base has $\frac{1}{4}$ the area of that of the pyramid and its height is half as great. The solid below the plane will then have $\frac{7}{8}$ the volume of the entire pyramid and the ratio of the two volumes is $\frac{1}{7}$.

553 *7. The volume of the complete pyramid which is 60 feet tall is 320 cubic feet. The base of the smaller pyramid is 30 feet above the ground so the part of the 60 foot pyramid to be included contains $\frac{7}{8} \cdot 320$ or 280 cubic feet (see Problem 6). The small pyramid capping the monument has volume $\frac{1}{3} \cdot 4 \cdot 2$ or about 2.7 cubic feet. Hence, the volume of the obelisk in cubic feet is approximately 282.7.

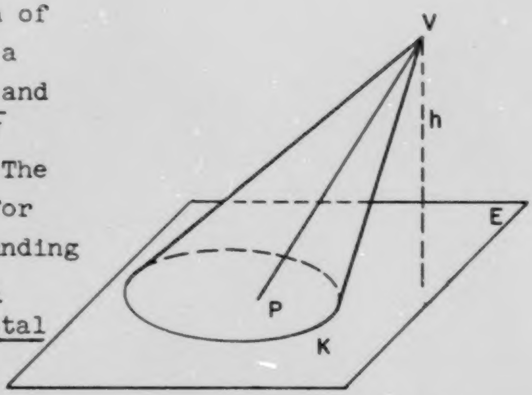
*8. Given a plane containing two regions and a line. If for every line which intersects the regions and is parallel to the given line the two intersections have equal lengths, then the two regions have the same area. Various examples are possible. Here is one:



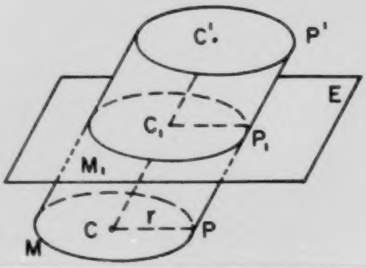
553 Here is a formal definition of circular cylinder, and associate terms. Let E_1 and E_2 be two parallel planes, L a transversal, and K a circular region in E_1, which does not intersect L. For each point P of K, let $\overline{PP'}$ be a segment parallel to L with P' in E_2. The union of all such segments is called a circular cylinder. K is the lower base, or just the base, of the cylinder. The set of all points P', that is, the part of the cylinder that lies in E_2, is called the upper base. Each segment $\overline{PP'}$ is called an element of the cylinder. (Note we did not introduce the term element in defining prism.) The distance h between E_1 and E_2 is the altitude of the cylinder. If L is perpendicular to E_1 and E_2 the cylinder is a right cylinder. Let M be the bounding circle of K and C the center of M. The union of all the elements $\overline{PP'}$ for which P belongs to M is called the lateral surface of the cylinder. The total surface is the union of the lateral surface and the bases. The element CC' determined by the center of M is the axis of the cylinder. Cross-sections are defined for cylinders exactly as for prisms.



55^4 Here is a formal definition of circular cone, and associate terms. Let K be a circular region in a plane E, and V a point not in E. For each point P in K there is a segment $\overline{PV}$. The union of all such segments is called a *circular cone* with *base* K and *vertex* V. Each segment $\overline{PV}$ is an *element* of the cone. The union of all elements $\overline{PV}$ for which P belongs to the bounding circle of K is the *lateral surface* of the cone. The *total surface* is the union of the lateral surface and the base. The distance h from V to E is the *altitude* of the cone. If the center of the base circle is the foot of the perpendicular from V to E, the cone is a *right* circular cone.



555 A formal proof of Theorem 16-11 is somewhat involved - we present a basis for a formal proof. Let M be the circle which bounds the base of the cylinder. Let C be the center of M and r its radius. Let E be the sectioning plane, and C_1 its *intersection* with the element $\overline{CC'}$ of the cylinder. Then the intersection of E with the lateral surface of the cylinder is the circle M_1 in E with center C_1 and radius r.



555 To prove this we must show that:

(a) Any point P_1 common to E and the lateral surface lies on M_1.

(b) Any point P_1 of circle M_1 is common to the lateral surface and E.

Proof of (a): Let P_1 be common to the lateral surface and E. Then P_1 lies on an element $\overline{PP'}$ where P is on circle M (by definition of lateral surface). Then $\overline{PP'} \parallel \overline{CC'}$, since any two elements of a cylinder are parallel. And $\overline{P_1C_1} \parallel \overline{PC}$ by Theorem 10-1. Thus, PP_1C_1C is a parallelogram and $P_1C_1 = PC = r$. That is P_1 lies on circle M_1.

Proof of (b): Let P_1 be a point of circle M_1. (Note P_1, P and P' are defined differently than in (a)). Let $\overleftrightarrow{P_1P}$ be parallel to $\overleftrightarrow{C_1C}$ and meet the base plane in P. Then $\overline{P_1C_1} \parallel \overline{PC}$ by Theorem 10-1 and PP_1C_1C is a parallelogram as above. Thus $PC = P_1C_1 = r$, so that P lies on circle M. Then P determines an element $\overline{PP'}$ and $\overleftrightarrow{PP'} \parallel \overleftrightarrow{CC_1}$. Since, $\overleftrightarrow{PP_1} \parallel \overleftrightarrow{CC_1}$, we see that $\overleftrightarrow{PP_1}$ and $\overleftrightarrow{PP'}$ coincide and P_1 lies on $\overleftrightarrow{PP'}$. From the diagram P_1 lies on $\overline{PP'}$. Thus, P_1 is on the lateral surface. Since P_1 is in E, the proof of (b) is complete.

Since M_1 bounds the cross-section, we have shown that the cross-section is a circular region. It remains to show it is congruent to the base. This is a relatively simple matter as outlined in the text.

555 Theorem 16-12 is immediate from Theorem 16-11, since the cross-section and the base are congruent circular regions.

555 In Theorem 16-13 the proof that the cross-section is a circular region is somewhat similar to that of Theorem 16-11. First one would prove that the intersection of the plane and the lateral surface is a circle.

In the diagram for Theorem 16-13, P is the center of the base circle and W is a point on it. Q and R are the intersections of the elements $\overline{PV}$ and $\overline{WV}$ with the sectioning plane.

The reasons in the proof of Theorem 16-13 are:

(1) The A.A. Similarity Theorem and the definition of similar triangles.

(2) $\overline{QR} \parallel \overline{PW}$ so that $\Delta\, VQR \sim \Delta\, VPW$. Then

$$\frac{QR}{PW} = \frac{VQ}{VP} = \frac{k}{h}.$$

(4) $\dfrac{\text{area of circle } Q}{\text{area of circle } P} = \dfrac{\pi QR^2}{\pi PW^2} = \left(\dfrac{QR}{PW}\right)^2 = \left(\dfrac{k}{h}\right)^2.$

557 Just as in proving Theorem 16-7 on the volume of a prism, consider a rectangular parallelepiped with the same altitude and base area as the given cylinder, and with its base coplanar with the base of the cylinder. Apply Cavalieri's Principle.

557 To prove Theorem 16-15 proceed as in Theorem 16-10. Take a triangular pyramid of the same altitude and base area as the cone and with its base coplanar with the base of the cone. Apply Cavalieri's Principle.

Problem Set 16-4

557 1. $V = \frac{1}{3}(9 \cdot \pi) \cdot 4 = 12\pi.$

2. The number of gallons is $\frac{\pi \cdot 14^2 \cdot 30}{3 \cdot 231} = \frac{22 \cdot 14 \cdot 14 \cdot 30}{7 \cdot 3 \cdot 3 \cdot 7 \cdot 11}$ $= \frac{80}{3} = 26\frac{2}{3}$. (The factors of 231 are $3 \cdot 7 \cdot 11$. By using $\frac{22}{7}$ the computation can be simplified by reducing fractions.)

3. Subtract the volume of the inner cylinder from that of the outer. This gives

$$16\pi(2.8)^2 - 16\pi(2.5)^2$$

or $16\pi(2.8^2 - 2.5^2) = 16\pi(2.8 - 2.5)(2.8 + 2.5)$

$= 16\pi(.3)(5.3) = 80$ approximately.

Approximately 80 cubic inches of clay will be needed.

4. The ratio of the volumes is the cube of the ratio of the altitudes, so

$$\frac{V_2}{V_1} = (\frac{2}{5})^3 = \frac{8}{125} = .064.$$

Hence $V_2 = .064 \times 27 = 1.73$ approx.

558 5. Let r be the radius of the base of the first can and h be its height. Then the radius of the second can is $2r$ and its height is $\frac{h}{2}$. Then

Volume of first can $= \pi r^2 h$.

Volume of second can $= \pi(2r)^2 \cdot \frac{h}{2} = 2\pi r^2 h$.

Since the volume of the second can is twice that of the first, and the cost is twice the cost of the first, neither is the better buy.

558 6. The volume of the pyramid is $\frac{20^2 \cdot 36}{3} = 4800$.

The radius of the base of the cone is half the diagonal of the square, or $10\sqrt{2}$.

The area of the base of the cone is $\pi(10\sqrt{2})^2 = 200\pi$,

and the volume of the cone is $\frac{200\pi \cdot 36}{3} = 2400\pi = 7{,}536$ approximately.

7. Let the radius of the base of each cylinder be r and the altitude be h. Then the volume of the cone in Figure 1 is $\frac{\pi r^2 h}{3}$. The volume of the two cones in Figure 2 is $2(\frac{\pi r^2}{3} \cdot \frac{h}{2}) = \frac{\pi r^2 h}{3}$.

The volumes are the same.

No, since the sum of altitudes would be the same as the altitude of cone in Figure 1.

8. $\pi r^2 h - \frac{1}{3}\pi r^2 h = \frac{2}{3}\pi r^2 h$.

559 *9. The volume of the frustum is the difference of the volumes of two pyramids. Hence their heights must be found. If x represents the height of the upper pyramid

$$\frac{x}{x + 8} = \frac{4}{6}, \text{ from which}$$

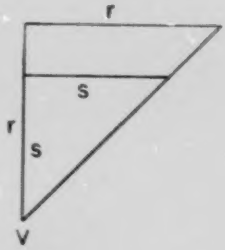
$$x = 16 \text{ and } x + 8 = 24.$$

$$\frac{1}{3}\pi 6^2 \cdot 24 - \frac{1}{3}\pi \cdot 4^2 \cdot 16 = \frac{608\pi}{3}.$$

The volume is approximately 636 cubic inches.

559 To prove Theorem 16-16 we show that the sphere and the solid bounded by the cylinder and the two cones have the same volume by Cavalieri's Principle. Then we can find the volume of the sphere by subtracting the volumes of the two cones from the volume of the cylinder.

560 The answer to the "Why?" is as follows: Consider one of the cones. Since the altitude of the cylinder is $2r$ the altitude of the cone is r. Also the radius of the base circle of the cone is r. Therefore, an isosceles right triangle is formed by the altitude, the radius, and a segment on the surface of the cone joining the vertex V to a point on the base. Any line parallel to the radius, intersecting the other two sides of this triangle, will form a triangle similar to the original one. Hence, the cross-section of the cone at a distance s from the vertex will be a circular region with radius s; and s will be the inner radius of the section of the solid.

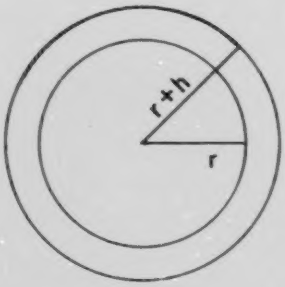


561 The argument of Theorem 16-17 should not be considered a formal proof, but an interesting example of mathematical reasoning based on a rather plausible assumption, namely, that S, the surface area of the inner sphere is the limit of $\frac{V}{h}$ as h approaches zero, where V is the volume of the spherical shell and h is its thickness. (We must either define the area of a surface or introduce some postulate concerning it, if we want to reason about it mathematically.) To justify intuitively that hS is approximately the volume of the spherical shell, we may consider it cut open and flattened out like a pie crust to form a thin, nearly flat,

cylinder. Then S becomes the area of the base of the cylinder and h its height, so that, its volume is hS. (Actually such a process would involve distortion and the volume of the shell would be slightly greater than hS.)

In the course of reasoning when we say $\frac{V}{h} \longrightarrow S$ as h grows smaller and smaller (or h approaches zero) we mean precisely the following: Let h take as its successive values an endless sequence of positive numbers, $h_1, h_2, \ldots, h_n, \ldots$ which approach zero (for example, $\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, \ldots$). Then since V is determined by the value assigned to h, V will take on a corresponding sequence of values, $V_1, V_2, \ldots, V_n, \ldots$, We assert that the sequence of quotients $\frac{V_1}{h_1}, \frac{V_2}{h_2}, \ldots, \frac{V_n}{h_n}, \ldots$ will approach the fixed number S.

You may better appreciate this method if we apply it in a simpler case to derive the formula for the circumference of a circle. Consider a circular ring with fixed inner radius r, outer radius $r + h$ and inner circumference C. The area A of the ring is approximately hC (it can be flattened out to approximately a thin rectangle) and $\frac{A}{h}$ is approximately C. As the ring gets thinner and thinner the approximation gets better and better, that is, $\frac{A}{h} \longrightarrow C$ as $h \longrightarrow 0$. But $A = \pi(r + h)^2 - \pi r^2 = 2\pi rh + \pi h^2$ so that



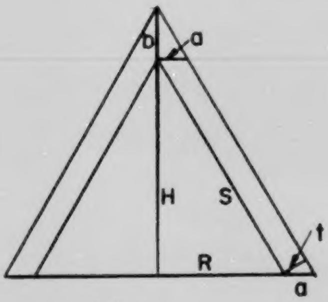
$$\frac{A}{h} = 2\pi r + \pi h.$$

Now let $h \longrightarrow 0$. Then $\frac{A}{h} \longrightarrow 2\pi r$. But C is the value which $\frac{A}{h}$ approaches. Therefore, $C = 2\pi r$.

A corresponding derivation for the area of the lateral surface of a cylinder is similarly handled (see Problem 11 of Problem Set 16-5).

For the lateral surface of a right circular cone it is somewhat more complicated and is given in detail below.

Derivation of Lateral Area of a Right Circular Cone.



The figure shows a vertical section of a right circular cone of base radius R, altitude H, and slant height S. It is covered with a layer of paint of thickness t. From similar triangles we have

$$\frac{t}{H} = \frac{a}{S}, \quad \text{and} \quad \frac{b}{H} = \frac{a}{R}.$$

Hence,

$$t = \frac{Ha}{S}, \qquad b = \frac{Ha}{R}.$$

The volume of the paint is

$$\begin{aligned} V &= \tfrac{1}{3}\pi(R + a)^2(H + b) - \tfrac{1}{3}\pi R^2 H \\ &= \tfrac{1}{3}\pi(2RHa + Ha^2 + R^2b + 2Rab + a^2b) \\ &= \tfrac{1}{3}\pi(2RHa + Ha^2 + RHa + 2Ha^2 + \tfrac{H}{R}a^3) \\ &= \tfrac{1}{3}\pi(3RHa + 3Ha^2 + \tfrac{H}{R}a^3) \\ &= \pi Ha(R + a + \tfrac{a^2}{3R}) \end{aligned}$$

We assume that the lateral area A is the limit approached by $\frac{V}{t}$ as t approaches zero. From above,

$$\frac{V}{t} = \pi S(R + a + \frac{a^2}{3R}).$$

As t gets very small so does a get very small, and so $\frac{V}{t}$ approaches the limit πSR.

Problem Set 16-5

562 1. Surface area: $4\pi 16$. Approximately 201.
Volume: $\frac{4}{3}\pi 64$. Approximately 268.

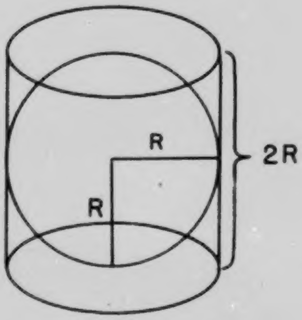
2. $\frac{4}{1}$; $\frac{8}{1}$. $\frac{9}{1}$; $\frac{27}{1}$.

3. $\frac{4\cdot 22\cdot 7\cdot 7\cdot 7\cdot 12\cdot 12\cdot 12}{3\cdot 7\cdot 7\cdot 3\cdot 11} = 10{,}752.$

The tank will hold approximately 10,752 gallons.

4. The area of a hemisphere is one-half the area of a sphere, or $2\pi r^2$. Since the area of the floor is πr^2, twice as much paint is needed for the hemisphere, or 34 gallons.

5. Volume of cylinder is $\pi R^2 \cdot 2R = 2\pi R^3$. $\frac{2}{3}(2\pi R^3) = \frac{4}{3}\pi R^3$, which is the formula for the volume of the sphere.

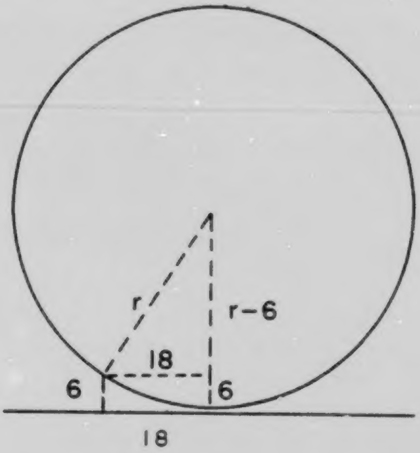


6. Since $r = 1$, the volume of the ice cream is $\frac{4}{3}\pi$ and the volume of the cone is $\frac{5}{3}\pi$. Therefore, the cone will not overflow.

563 7. a. The volume of a cube of edge s is s^3; the volume of a cube of edge $4s$ is $(4s)^3$ or $64s^3$. Hence, the ratio of the volumes is 64 to 1.

b. If R and $4R$ are radii of the moon and the earth the volumes have the ratio $\frac{\frac{4}{3}\pi R^3}{\frac{4}{3}\pi(4R)^3} = \frac{1}{64}$.

563 8. The altitude of the cone is r plus the hypotenuse of a $30^\circ - 60^\circ$ right triangle with short side r. So the altitude is $3r$. Using a right triangle determined by the altitude of the cone and a radius of the base, the radius of base of the cone is $r\sqrt{3}$, so the area of the base is $3\pi r^2$. The volume of the cone is therefore $\frac{1}{3} \cdot 3\pi r^2 \cdot 3r = 3\pi r^3$.



*9. Let r be the radius of the tank in feet.

$r^2 = 18^2 + (r - 6)^2.$

$r^2 = 324 + r^2 - 12r + 36.$

$12r = 360.$

$r = 30.$ The radius is 30 feet. Using $V = \frac{4}{3}\pi r^3$, the volume of the tank in cubic feet is $\frac{4\pi \cdot 30^3}{3}$.

Converting this to cubic inches, finding the number of gallons contained, and dividing by 10,000, the number of hours a tank full will last is $\frac{4 \cdot 22 \cdot 27000 \cdot 1728}{3 \cdot 7 \cdot 231 \cdot 10000}$ or about 85 hours.

*10. Let V be the original volume and R the original radius, v the new volume and r the new radius.

Then
$$\frac{V}{v} = \frac{2}{1} = \frac{\frac{4}{3}\pi R^3}{\frac{4}{3}\pi r^3} = \frac{R^3}{r^3}.$$

Therefore, $\frac{R^3}{r^3} = \frac{2}{1}$ or $\frac{R}{r} = \frac{\sqrt[3]{2}}{1}$.

Hence, $r = \frac{R}{\sqrt[3]{2}} = \frac{\sqrt[3]{4}\,R}{2}$.

Since, $\sqrt[3]{4}$ is approximately 1.6,

r is approximately $\frac{4}{5}$ R.

563 *11. Let V be the volume of a cylindrical shell, S the lateral area of the cylinder, and h the thickness of the shell. Then $\frac{V}{h} \longrightarrow S$ as h gets smaller and smaller. By Theorem 16-14 we know that

$$\begin{aligned} V &= \pi(r + h)^2a - \pi r^2a \\ &= 2\pi rha + \pi h^2a. \end{aligned}$$

Therefore,

$$\frac{V}{h} = 2\pi ra + \pi ha.$$

Since, $h \longrightarrow 0$, $\pi ha \longrightarrow 0$ and $\frac{V}{h} \longrightarrow 2\pi ra$.

Hence, $S = 2\pi ra$.

Review Problems

564 1. a. rhombus, 120, 60.

b. 8. c. $32\sqrt{3}$.

2. 61 approx. $\frac{4}{3}\pi \cdot \frac{5}{2} \cdot \frac{5}{2} \cdot \frac{5}{2} - \frac{4}{3}\pi \cdot 1 \cdot 1 \cdot 1.$

$$\frac{4}{3}\pi\left(\frac{125}{8} - 1\right) = \frac{4}{3}\pi \cdot \frac{117}{8} = \frac{39\pi}{2} = 61.26$$

3. 19 approx. $\frac{1}{3}\pi \cdot 25 \cdot h = 500.$

$$h = \frac{60}{\pi} = 19 \text{ approx.}$$

4. 48 square inches. $\frac{1}{3}B \cdot 12 = 432.$

$$B = 108.$$

If A is the area of the cross-section,

$$\frac{108}{A} = \frac{144}{64}.$$

$$A = 48.$$

5. The volume of the first is half the volume of the second.

564 6. 4872 approx. $\pi \cdot 144 \cdot 20 - \frac{4}{3}\pi \cdot 10 \cdot 10 \cdot 10 =$

$\pi(2880 - \frac{4000}{3}) = \frac{4640}{3}\pi = 4872$ approximately.

7. $$\frac{V_s}{V_c} = \frac{\frac{4}{3}\pi r^3}{\pi r^2 \cdot 2r} = \frac{2}{3}.$$

565 *8. The volume of the solid equals the volume of the large cone decreased by the sum of the cylinder and the small, upper cone. Let h be the altitude of the small cone. Then $15 - h$ is the altitude of the cylinder. Since the cones are similar,

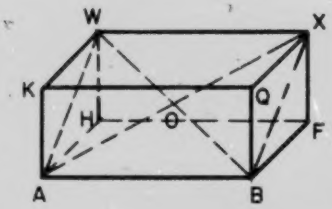
$$\frac{h}{15} = \frac{3}{8} \quad \text{and} \quad h = \frac{45}{8}.$$

Hence, $V = \frac{1}{3}\pi 64 \cdot 15 - (\pi 9 \cdot \frac{75}{8} + \frac{1}{3}\pi 9 \cdot \frac{45}{8})$

$$= 320\pi - \frac{810\pi}{8} = \frac{875\pi}{4} = 687.5 \text{ approx.}$$

9. A diagonal of a parallelogram divides it into congruent triangles. Therefore, by Theorem 16-8 the pyramid is divided into two pyramids of equal volume.

*10. In the rectangular parallelepiped, diagonals $\overline{AX}$ and $\overline{WB}$ of rectangle ABXW are congruent and bisect each other at O. Similarly, diagonals $\overline{KF}$ and $\overline{HQ}$ bisect each other at O'. By considering the intersection of $\overline{KF}$ and $\overline{WB}$, it is evident that O' = O. Therefore, all four diagonals bisect each other at O. Since the diagonals are congruent, it follows that O is equidistant from each of the vertices, and is the center of the required sphere.



Illustrative Test Items for Chapter 16

A. Indicate whether each statement is true or false.

1. A plane section of a triangular prism may be a region whose boundary is a parallelogram.
2. A plane section of a triangular pyramid may be a region whose boundary is a parallelogram.
3. The volume of a triangular prism is half the product of the area of its base and its altitude.
4. In any pyramid a section made by a plane which bisects the altitude and is parallel to the base has half the area of the base.
5. Two pyramids with the same base area and the same volume have congruent altitudes.
6. The volume of a pyramid with a square base is equal to one-third of its altitude multiplied by the square of a base edge.
7. The area of the base of a cone can be found by dividing three times the volume by the altitude.
8. The volume of a sphere is given by the formula $\frac{1}{6}\pi d^3$ where d is its diameter.
9. All cross-sections of a rectangular parallelepiped are rectangles.
10. A cross-section of a circular cone is congruent to the base.
11. Two prisms with congruent bases and congruent altitudes are equal in volume.
12. In a sphere of radius 3, the volume and the surface area are expressed by the same number.

13. The area of the cross-section of a pyramid that bisects the altitude is one-fourth the area of the base.

14. The diagonal of a rectangular parallelepiped is $\frac{1}{3}$ the sum of the three dimensions of the parallelepiped.

15. In a right circular cone the segment joining the vertex with the center of the base is the altitude of the cone.

B. 1. A school room is 22 feet wide, 26 feet long and 12 feet high. If there should be an allowance of 200 cubic feet of air space for each person in the room, and if there are to be two teachers in the room, how many pupils may there be in a class?

2. A 24 inch length of wire is used to form a model of the edges of a cube. How long a wire is needed to form the edges of a second cube, if an edge of the second is double an edge of the first? What is the ratio of the surface areas of the two cubes? Of their volumes?

3. A square 6 inches on a side is revolved about one diagonal. Give the volume of the solid thus "generated".

4. If a right circular cone is inscribed in a hemisphere such that both have the same base, find the ratio of the volume of the cone to the volume of the hemisphere.

C. 1. If a cone and a cylinder have the same base and the same altitude, the volume of the cylinder is _____ times the volume of the cone.

2. If the area of one base of a cylinder is 24 square inches, the area of the other base is ______ square inches.

3. In a circular cylinder with radius 5 and altitude 6, the area of a cross-section one-half inch from the base is ______ π.

4. In a circular cone with radius 5 and altitude 6, the area of a cross-section at a distance 2 from the vertex is _____.

5. The area of the base of a pyramid with altitude 12 inches is ______ times the area of a cross-section 2 inches from the base.

6. If the area of a cross-section of a pyramid is $\frac{1}{4}$ the area of the base, this cross-section of the pyramid divides the altitude of the pyramid into two segments whose ratio is ____ to ____.

7. The base of a pyramid is an equilateral triangle whose perimeter is 12. If the altitude is 10, the volume of the pyramid is _____.

8. The base of a prism is a parallelogram with sides 10 and 8 determining a 30° angle. If the altitude of the prism is 14, the volume is ______.

9. If the dimensions of a rectangular parallelepiped are 3, 5, 6, the length of a diagonal is _______; the total surface area is _______; and the volume is _____.

10. If the diameter of a sphere is 12, the volume of the sphere is _______, the area of a great circle is ______, and the area of the sphere is _______.

Answers

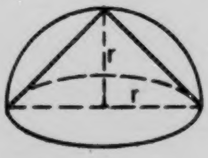
A. 1. T,
2. T,
3. F,
4. F,
5. T,
6. T,
7. T,
8. T,
9. T,
10. F,
11. T,
12. T,
13. T,
14. F,
15. T.

B. 1. 32 pupils. $\frac{22 \times 26 \times 12}{200} = 34.3$.

2. 48 inches. $\frac{1}{4}$. $\frac{1}{8}$.

3. $36\pi\sqrt{2}$. The solid consists of two right circular cones with a common base having $r = h = 3\sqrt{2}$.

4. $\frac{V_c}{V_H} = \frac{\frac{1}{3}\pi r^2 \cdot r}{\frac{1}{2} \cdot \frac{4\pi r^3}{3}} = \frac{1}{2}$.



C. 1. 3.
2. 24.
3. 25π.
4. $\frac{25\pi}{9}$.
5. $\frac{36}{25}$.
6. 1, 1.
7. $\frac{40\sqrt{3}}{3}$.
8. 560.
9. $\sqrt{70}$, 126, 90.
10. 288π, 36π, 144π.

Chapter 17

PLANE COORDINATE GEOMETRY

The inclusion of a chapter on analytic geometry in a tenth grade geometry course is a recent innovation. We introduced it at the end of the book for two reasons.

First, for flexibility in using the text. Some teachers may prefer to teach analytic geometry in the eleventh grade (or later) in order to do justice to this very important idea which shows the complete logical equivalence of synthetic geometry and high school algebra. They may feel that the tenth grade already is crowded with many essential things, and that to crowd it further does not do a service to the understanding of synthetic geometry as a mathematical system or of the analytic approach. On the other hand, some teachers may feel a sense of excitement over the opportunity to introduce students to analytic geometry, and may be grateful for a chance to communicate this excitement to their students at the expense of omitting some more conventional material.

Secondly, the analytic geometry was introduced at the end in order to do justice to both synthetic geometry and analytic geometry. If the student is to obtain a deep appreciation of the equivalence of Euclidean Geometry and classical algebra, he must understand these as separate disciplines. He already has spent much time in the study of algebra, and it does not seem desirable to fragment the treatment of synthetic geometry with the piecemeal introduction of analytic ideas - he may fail to grasp that there is an autonomous subject of geometry which is logically equivalent to the autonomous subject of algebra.

460 In fact, a surprising number of the concepts treated earlier in the book are necessary for analytic geometry. The most obvious of these concepts is that of the number scale, but much more than this is involved. The idea of plane separation is involved in distinguishing the location of points with positive coordinates and points with negative coordinates. The theory of parallels justifies the rectangular network used for graphs. Similarity is used in establishing the constant slope of a line. The Pythagorean Theorem forms the basis for the distance formula. The notion of a set of points satisfying certain conditions, which is basic in coordinate geometry, is treated synthetically in Chapter 14. These few examples will serve to illustrate the considerable background of concepts it is desirable for a student to have before beginning a careful treatment of analytic geometry.

567 The history of geometry, like the history of all of mathematics, is a fascinating story. When one knows the history of a subject, he can better appreciate the years of development necessary to put it into the form we know it today. Since the development of analytic geometry was a major break-through in mathematical thought at the time Descartes discovered it, students might be interested in the history of its development and discovery, just as they might be interested in the history of synthetic geometry. Suggest to them the title of an available book on the history of mathematics. (An excellent bibliography has recently been published by the National Council of Teachers of Mathematics. Write for the pamphlet "The High School Mathematics Library", by William L. Schaaf. Address: NCTM, 1201 Sixteenth Street, N.W., Washington 6, D.C.)

The idea of translating between algebra and geometry can be used by the teacher as a means of organizing a cumulative summary of the chapter. The students can be asked to keep a geometry-algebra dictionary like the following.

Geometry	Algebra
A point P in a plane	An ordered pair of numbers (x,y).
The end-points of a segment $\overline{P_1P_2}$.	(x_1,y_1) and (x_2,y_2).
The slope of $\overline{P_1P_2}$.	The number $m = \frac{y_2 - y_1}{x_2 - x_1}$.
The distance P_1P_2.	The number $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
The mid-point of $\overline{P_1P_2}$.	$(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$.
A line.	The set of ordered pairs of numbers that satisfy some linear equation $Ax + By + C = 0$.
The intersection of two lines.	The common solution of two linear equations.
Two non-vertical lines are parallel.	$m_1 = m_2$.
Two non-vertical lines are perpendicular.	$m_1m_2 = -1$.

568 Notice that we now set up a coordinate system on each of two perpendicular lines, rather than on only one line, as we did in Chapter 2. This enables us to find the coordinates of the projections of any point on the two lines. We write these coordinates as an ordered pair (x,y).

We again have a one-to-one correspondence, this time between ordered pairs of real numbers and points in a plane. To each ordered pair of real numbers there corresponds one and only one point in the plane, and to each point in the plane there corresponds one and only one ordered pair of real numbers.

Sections 17-2 and 17-3 cover material that is familiar to most students, and classes should move on as quickly as possible. If students already know the terms abscissa and ordinate, there is no reason to object to their use of these words. The terms are superfluous, however, and need not be introduced by you.

Problem Set 17-3

57^4 1. "Cartesian" is used to honor the discoverer, Descartes.

2. (0,0).

3. -3.

4. The origin, or (0,0).

5. (2,1) and (2,0).

6. a. IV. c. I.
   b. II. d. III.

7. One of the coordinates must be 0.

8. D, B, C, A.

9. C, A, D, B.

10. a. II. e. IV.
    b. I. f. I.
    c. IV. g. II.
    d. III. h. III.

575 *12. a. y-axis, x-axis, z-axis.

b. xz-plane, yz-plane, xy-plane.

c. 4, 2, 3.

576 When we define the slope of a line segment to be the quotient of the difference between pairs of coordinates, there is no need to introduce the notion of directed distance, but it is absolutely necessary to put the coordinates of the two points (x_1,y_1) and (x_2,y_2) in the proper position in the formula. That is $m = \frac{y_2 - y_1}{x_2 - x_1}$ cannot be used as $m = \frac{y_2 - y_1}{x_1 - x_2}$ although $m = \frac{y_1 - y_2}{x_1 - x_2}$ is also correct. Notice that in finding the slope of $\overline{AB}$ it doesn't matter which point is labeled P_1 and which one is labeled P_2.

578-579 It is important to note here that RP_2 and P_1R are positive numbers and we have to prefix the minus sign to the fraction $\frac{RP_2}{P_1R}$ if the slope is negative. However, the formula defining the slope of a segment will give the slope m as positive or negative without prefixing any minus sign.

For the Case (1) if $m > 0$, then $m = \frac{RP_2}{P_1R}$, $RP_2 = y_2 - y_1$ and $P_1R = x_2 - x_1$. For the Case (2) if $m < 0$, then $m = -\frac{RP_2}{P_1R}$, $RP_2 = y_2 - y_1$ and $P_1R = x_1 - x_2$. Therefore Case (2) becomes $m = -\frac{y_2 - y_1}{x_1 - x_2}$ which is equivalent to $m = \frac{y_2 - y_1}{x_2 - x_1}$.

Problem Set 17-4

580 1. a. 7. b. -1. c. y_1.

2. a. 6. b. -3. c. x_1.

3. a. 2. b. 2. c. 3.

d. The two points in each part have the same y-coordinate.

581 e. If two points in a plane have the same y-coordinate, then the distance between them is the absolute value of the difference of their x-coordinates.

f. No.

4. a. 3. b. 2. c. 4.

d. $|y_1 - y_2|$ or $|y_2 - y_1|$.

e. The two points in each part have the same x-coordinate.

f. If two points in a plane have the same x-coordinate, the distance between them is the absolute value of the difference of their y-coordinates.

5. (2,3); (-1,-5); (3,-1).

6. PA = 2, QA = 2.
PB = 5, QB = 3.
PC = 7, QC = 3.

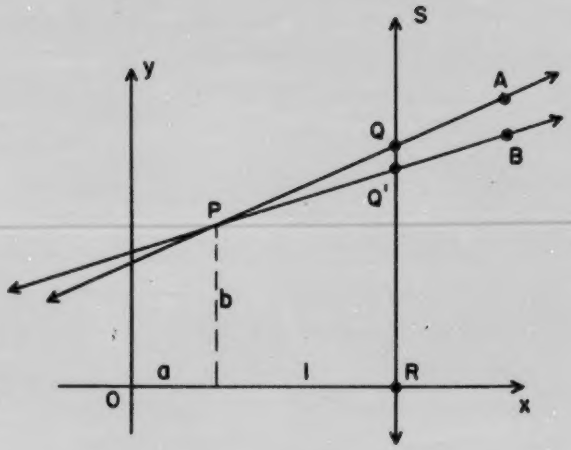
7. $-1, \frac{3}{5}, \frac{3}{7}$.

8. $\frac{1}{15}$.

582 9. a. $\frac{1}{3}$. e. $-\frac{15}{8}$.
b. -3. f. $\frac{8}{15}$.
c. $\frac{7}{4}$. g. -1.
d. $\frac{3}{4}$. h. -3.

10. a. 6. b. 4.5.

*11. First assume that $\overleftrightarrow{PA}$, $\overleftrightarrow{PB}$ have the same slope m. Let $P = (a,b)$, $R = (a + 1,0)$. Let $\overleftrightarrow{RS}$ be perpendicular to the x-axis. Neither $\overleftrightarrow{PA}$ nor $\overleftrightarrow{PB}$ is perpendicular to the x-axis, hence, neither $\overleftrightarrow{PA}$ nor $\overleftrightarrow{PB}$ is parallel to $\overleftrightarrow{RS}$. Let $\overleftrightarrow{PA}$, $\overleftrightarrow{PB}$ intersect $\overleftrightarrow{RS}$ in Q, Q', respectively. Let $Q = (a + 1,c)$, $Q' = (a + 1,c')$



Then $\frac{c - b}{1} = m = \frac{c' - b}{1}$.

Whence, $c = c'$ and hence $Q = Q'$. Hence, $\overleftrightarrow{PA} = \overleftrightarrow{PB}$ (by Postulate 2).

The converse has already been proved (Theorem 17-1). Hence, if $\overleftrightarrow{PA}$, $\overleftrightarrow{PB}$ have different slopes, then P, A, B cannot be collinear.

12. a. Yes. b. No.

583 13. a. -1. b. $-\frac{3}{2}$. c. $\frac{a - b}{2b}$.

14. Slope of $\overleftrightarrow{AB}$ is $\frac{96}{96} = 1$. Slope of $\overleftrightarrow{BC}$ is $\frac{100}{100} = 1$. Point B is common. Therefore $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ coincide.

583 15. Slope of $\overleftrightarrow{AB}$ is $\frac{96}{96} = 1$; slope of $\overleftrightarrow{CD}$ is $\frac{1}{1} = 1$. We are tempted to say that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, but we must make sure that they are actually two different lines. We test by finding the slope of $\overleftrightarrow{AC}$, which is $\frac{101}{101} = 1$. Hence, $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ must coincide so that C is on $\overleftrightarrow{AB}$ and the lines can't be parallel. It follows that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ coincide.

16. Draw the segment which joins (4,3) and the origin; any other segment through the origin lying on the line determined by this segment will also suffice.

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583 The information concerning slopes of parallel and perpendicular lines constitutes a very important principle for the solving of geometric problems analytically. For instance, if a student were asked to show that two non-vertical lines were parallel, he would have to show that their slopes were equal; to show that a pair of oblique lines were perpendicular would require that he establish the slopes to be negative reciprocals of each other. Note that to show two segments parallel it is not sufficient to show they have the same slope; it is necessary to show also that the segments are not collinear (see Problems 11 and 15 of Problem Set 17-4).

585 To show why $\Delta\ PQR \cong \Delta\ Q'PR'$ we first show that $\angle\ Q'PR'$ is complementary to $\angle\ QPR$. This follows from $m\angle\ Q'PR' + m\angle\ Q'PQ + m\angle\ QPR = 180$ and $m\angle\ Q'PQ = 90$. Therefore $\angle\ Q'PR' \cong \angle\ PQR$ and $\angle\ PQ'R' \cong \angle\ QPR$. Since $PQ = PQ'$, the triangles are congruent by A.S.A.

In the converse we use S.A.S. to show $\Delta\ PQR \cong \Delta\ Q'PR'$ By construction, $R'P = RQ$ and $\angle\ R$ and $\angle\ R'$ are right angles. We get $R'Q' = PR$ as follows: $m = \frac{RQ}{PR}$ and $m' = -\frac{R'Q'}{R'P}$. Then $m' = -\frac{1}{m}$ becomes $-\frac{R'Q'}{R'P} = -\frac{PR}{RQ}$, and since the denominators are equal we have $R'Q' = PR$.

Finally, we get $\angle Q'PQ$ a right angle by using the fact that $\angle Q'PR$ is an exterior angle of $\Delta PQ'R'$ and that $\angle QPR \cong \angle PQ'R'$.

Note that Theorem 17-2, and some theorems which follow, are stated after the proof rather than before. In this way, the full theorem seems to be a result of the discussion pertinent to the topic being considered.

Problem Set 17-5

586 1. Slope $\overline{AB} = \frac{3}{2}$; slope $\overline{CD} = \frac{3}{2}$; hence, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ or $\overleftrightarrow{AB} = \overleftrightarrow{CD}$. Slope $\overline{AC} = \frac{-4}{5}$, hence, A, B, C, are not collinear. (See Problems 11 and 15 of Problem Set 17-4.) Hence, $\overleftrightarrow{AB} \neq \overleftrightarrow{CD}$, so that $\overline{AB} \parallel \overline{CD}$. Similarly, prove $\overline{BC} \parallel \overline{AD}$.

2. Slope of $\overline{AB} = -\frac{2}{3}$, slope of $\overline{CD} = -\frac{2}{3}$.
Slope of $\overline{BC} = -3$, slope of $\overline{DA} = -3$.
Therefore opposite sides are parallel and the quadrilateral is a parallelogram.

3. $L_1 \perp L_3$ and $L_2 \perp L_4$, by Theorem 17-3.

587 4. The second is a parallelogram, as can be shown from the slopes of $\overline{PQ}$, $\overline{RS}$, $\overline{QR}$, and $\overline{PS}$, which are respectively, $\frac{2}{3}$, $\frac{2}{3}$, $-\frac{1}{5}$, $-\frac{1}{5}$. The first is not a parallelogram since the slopes of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{AD}$ are respectively, 4, $\frac{1}{2}$, 5, and $\frac{3}{8}$.

5. a. Slope of $\overline{AB} = -\frac{2}{7}$.

Slope of $\overline{BC} = \frac{2}{9}$.

Slope of $\overline{AC} = 0$.

587 b. Slope of altitude to $\overline{AB} = \frac{7}{2}$.

Slope of altitude to $\overline{BC} = -\frac{9}{2}$.

The altitude to $\overline{AC}$ has no slope; it is a vertical segment.

6. Both $\overline{AB}$ and $\overline{CD}$ have the same slope, -1; $\overline{AC}$ has slope 0. Therefore $\overline{AB} \parallel \overline{CD}$. $\overline{AD}$ and $\overline{BC}$ have different slopes. Therefore the figure is a trapezoid. Diagonal $\overline{AC}$ is horizontal since its slope is 0. Diagonal $\overline{BD}$ is vertical. A vertical and a horizontal line are perpendicular.

7. The slope in each case is the same, $-\frac{1}{3}$; the slope of line joining $(3n,0)$ to $(6n,0)$ is 0. Hence, the given lines are parallel.

8. The slope of the first line is $\frac{b}{a}$. The slope of the second is $-\frac{a}{b}$. Since the negative reciprocal of $\frac{b}{a}$ is $-\frac{a}{b}$, the lines are perpendicular.

*9. Application of the slope formula shows that the slope of $\overline{XY}$ is $\frac{b}{a}$, and that of $\overline{XZ}$ is $-\frac{a}{b}$. By Theorem 17-3, $\overline{XY} \perp \overline{XZ}$. Hence, $\angle X$ is a right angle.

10. $\angle PQR$ will be a right angle if $\overline{PQ} \perp \overline{QR}$. $\overline{PQ}$ will be perpendicular to $\overline{QR}$ if their slopes are negative reciprocals; that is, if:

$$\frac{-6 - 2}{5 - 1} = -\frac{b - 5}{b + 6}$$

from which $b = -17$.

11. Slope $\overline{PQ} = \frac{-1}{a - 3}$; slope $\overline{RS} = \frac{-1}{b - 4}$; slope $\overline{QS} = 0$. If $\overleftrightarrow{PQ}$ were the same as $\overleftrightarrow{RS}$ these three slopes would have to be equal; but neither of the first two can be zero for any value of a or b.

If $\overline{PQ} \parallel \overline{RS}$ then $\frac{-1}{a - 3} = \frac{-1}{b - 4}$, whence, $a = b - 1$.

588 Notice that it would be impossible for us to develop the distance formula without the Pythagorean Theorem, which in turn rests upon the theory of areas, parallels, and congruence.

It might be instructive with a good class to have them derive the distance formula with P_1 and P_2 in various positions in the plane. In working with the distance formula, it does not matter in which order we take P_1 and P_2 in as much as we will be squaring the difference between coordinates. The distance formula holds even when the segment $\overline{P_1P_2}$ is horizontal or vertical.

Problem Set 17-6

590 1. a and b. AB = 1, AC = 3, AD = 4.5, BC = 4, BD = 3.5, CD = 7.5.

2. a. $|x_2 - x_1|$ or $\sqrt{(x_2 - x_1)^2}$.

   b. $|y_2 - y_1|$ or $\sqrt{(y_2 - y_1)^2}$.

3. a. 5.
   b. 5.
   c. 13.
   d. 25.
   e. 17.
   f. $\sqrt{2}$.
   g. 89.
   h. $5\sqrt{5}$.

4. a. $(y_2 - y_1)^2 + (x_1 - x_2)^2$.

   b. $x^2 + y^2 = 25$.

591 5. By the distance formula RS = 5, RT = $\sqrt{2}$ and ST = 5. Since ST = RS the triangle is isosceles.

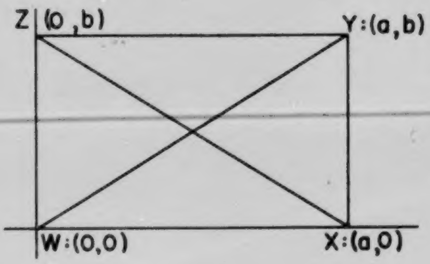
591 6. Δ DEF will be a right triangle with $\angle D$ a right angle only if $DE^2 + DF^2 = EF^2$. This is the case since $DE^2 = 5$, $DF^2 = 45$ and $EF^2 = 50$.

7. $AB = \sqrt{8} = 2\sqrt{2}$. $BC = \sqrt{72} = 6\sqrt{2}$. $AC = \sqrt{128} = 8\sqrt{2}$. Hence, $AB + BC = AC$, and therefore, from the Triangle Inequality, A, B, C, are collinear. It now follows from the definition of "between" that B is between A and C.

8. a. 7. b. 5.

9. a. (a,b).

b.



$$WY = \sqrt{(a - 0)^2 + (b - 0)^2} = \sqrt{a^2 + b^2}.$$

$$XZ = \sqrt{(0 - a)^2 + (b - 0)^2} = \sqrt{a^2 + b^2}.$$

Hence, $WY = XZ$.

*10. a. Let $A = (2,0,0)$, $B = (2,3,0)$. From the meaning of the x, y, and z-coordinates, $OA = 2$, $AB = 3$, and $BP = 6$. By the Pythagorean Theorem applied to Δ OAB, $OB^2 = 13$, then applied to Δ OBP, $OP^2 = 49$ and $OP = 7$. ($\overline{OP}$ may also be considered a diagonal of a rectangular block.)

b. Generalizing the procedure in part (a), the distance is $\sqrt{x^2 + y^2 + z^2}$.

c. $P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

[page 591]

592 The mid-point formula will prove to be very useful in the work which follows. This will be true, for example, when we are speaking of the medians of a triangle. If we know the coordinates of the vertices of a triangle, and apply the definition of a median, we can find the coordinates of the point in which the median intersects the opposite side. This will give us the coordinates of its end-points and enable us to find the length and slope of the median.

The proof of the mid-point formula is easily modified to hold for horizontal and vertical segments.

Problem Set 17-7

593 1. a. (0,6).

b. (-2.5,0).

c. (2,0).

d. (0,0).

e. (0.0).

2. a. (8,12).

b. (-5.5,-1.5).

c. $(\frac{5}{12},\frac{13}{80})$.

d. (1.58,1.11).

e. $(\frac{a+b}{2},\frac{c}{2})$.

f. $(\frac{s}{2},\frac{r}{2})$.

594 3. a. (4,2).

b. $-9 = \frac{13 + x}{2}$, $30 = \frac{19 + y}{2}$,

$x = -31$. $y = 41$.

The other end-point is at (-31,41).

4. $\overline{AC} \cong \overline{BD}$ since both have lengths $\sqrt{68}$ by the distance formula. $\overline{AC} \perp \overline{BD}$ since the slope of $\overline{AC}$ is 4 and the slope of $\overline{BD}$ is $-\frac{1}{4}$. These are negative reciprocals. $\overline{AC}$ and $\overline{BD}$ bisect each other since using the mid-point formula each has the mid-point (3,5).

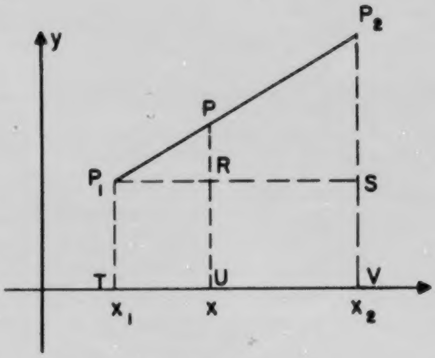
594 5. The mid-point X of $\overline{AB}$ is (3,2).
The mid-point Y of $\overline{BC}$ is (-1,3).
The mid-point Z of $\overline{CA}$ is (1,0).

By the distance formula $CX = \sqrt{37}$, $AY = \sqrt{52}$ or $2\sqrt{13}$, and $BZ = 5$.

6. By formula, the mid-points of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$ are W(0,1), X(-1,6), Y(4,6) and Z(5,1), respectively. $\overline{WX}$ has length $\sqrt{26}$ and slope -5. $\overline{YZ}$ also has length $\sqrt{26}$ and slope -5. $\overline{XY}$ has slope 0, hence, $\overleftrightarrow{WX} \neq \overleftrightarrow{YZ}$, so that, $\overline{WX} \parallel \overline{YZ}$. With the same two sides parallel and congruent the figure is a parallelogram.

7. By the mid-point formula the other end-point of one median is $(\frac{a}{2},\frac{3a}{2})$, and the other end of another median is $(\frac{-a}{2},\frac{3a}{2})$. By the slope formula, the slopes of these medians are 1 and -1. Since 1 is the negative reciprocal of -1, the medians are perpendicular.

8. From the similarity between $\Delta\ P_1PR$ and $\Delta\ P_1P_2S$, $P_1R = \frac{1}{3}\ P_1S$. Since $TU = P_1R$ and $TV = P_1S$, $TU = \frac{1}{3}\ TV$. In terms of coordinates $x - x_1 = \frac{1}{3}(x_2 - x_1)$, or $x = \frac{1}{3}(x_2 - x_1) + x_1$.



This can also be written $x = \dfrac{x_2 + 2x_1}{3}$. By a similar argument with $\overline{P_1P_2}$ projected into the y-axis,

$$y = \frac{y_2 + 2y_1}{3}.$$

[page 594]

Therefore the coordinates of P are

$$\left(\frac{x_2 + 2x_1}{3}, \frac{y_2 + 2y_1}{3}\right).$$

595 *9. a. Replacing $\frac{1}{3}$ by $\frac{r}{r + s}$ in the solution of the previous problem, if $x_2 > x_1$, we get

$$x = \frac{r}{r + s}(x_2 - x_1) + x_1,$$

from which, $x = \dfrac{r(x_2 - x_1) + x_1(r + s)}{r + s}$,

or $x = \dfrac{rx_2 + sx_1}{r + s}$.

If $x_2 < x_1$, a similar argument leads to the same result.

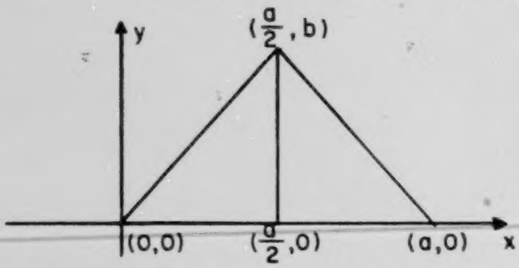
By a similar argument, with $\overline{P_1P_2}$ projected into the y-axis,

$$y = \frac{ry_2 + sy_1}{r + s}.$$

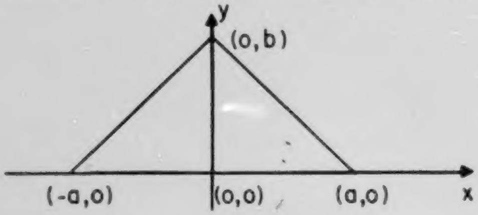
b. $x = \dfrac{3 \cdot 25 + 5 \cdot 5}{3 + 5} = 12.5;$

$y = \dfrac{3 \cdot 36 + 5 \cdot 11}{8} = 20.375.$

595 Although we may place our axes in any manner we desire in relation to a figure, there are advantages to be had by a clever choice. For instance, if we are given an isosceles triangle, we may place the axes wherever we wish, then use the properties of an isosceles triangle to determine the coordinates of the vertices. Suppose we place it like this:

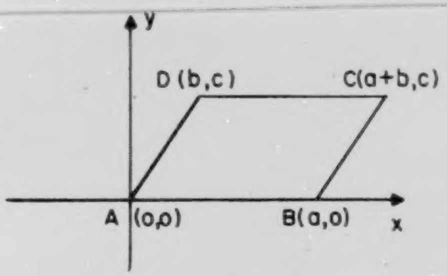


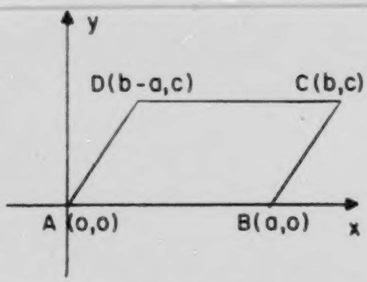
The student should be permitted to draw upon his knowledge of synthetic geometry and make use of the fact that the altitude to the base of an isosceles triangle bisects the base. Hence, the x-coordinate of the vertex should be half the x-coordinate of the end-point of the base that is not at the origin. On the other hand the y-coordinate of the vertex is not determined by the coordinates of the other vertices and is an arbitrary positive number. Suppose we place the axes like this with the vertex on the y-axis:



Then, since the altitude bisects the base, the lengths of the segments into which it divides the base are equal, and therefore the end-points of the base may be indicated by (a,0) and (-a,0).

There also are limits to what we can choose for coordinates. For parallelograms, we find that three vertices may be labeled arbitrarily, but the coordinates of the fourth vertex are determined by those of the other three. Naturally there is more than one way in which we may label a parallelogram. Below in the figure on the left the coordinates of points A, B, and D were assigned first. Then the coordinates of C were determined in terms of the coordinates of the other three points. In the figure on the right A, B and C were chosen first. Notice how the coordinates of D are given in terms of the other coordinates.





One word of CAUTION. The above discussion is based upon the fact that such things as isosceles triangles or parallelograms are given in the problem. If the problem is to prove that a quadrilateral is a parallelogram or that a triangle is isosceles, then we cannot assume such properties to be true, and must establish, as part of the exercise, sufficient properties to characterize the figure.

If class time is limited, the end of Problem Set 17-8 would provide a satisfactory conclusion to the coordinate geometry work. The balance of the chapter could be covered in later courses.

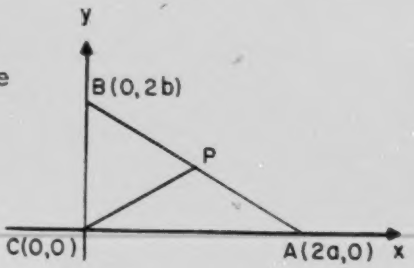
Problem Set 17-8

598 1. $DB = \sqrt{(a - 0)^2 + (0 - b)^2} = \sqrt{a^2 + b^2}$.

$AC = \sqrt{(a - 0)^2 + (b - 0)^2} = \sqrt{a^2 + b^2}$.

Therefore, $DB = AC$.

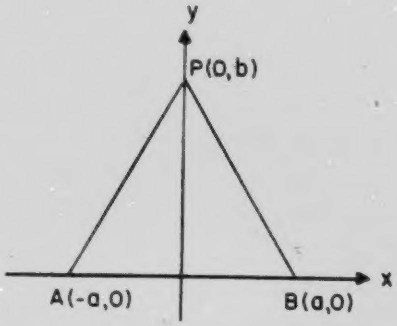
2. Locate the axes along the legs of the triangle as shown. By definition of midpoint $PA = PB$. Therefore $P = (a,b)$. It must be shown that $PA = PC$ (or that $PB = PC$). By the distance formula



$$PA = \sqrt{(2a - a)^2 + (0 - b)^2} = \sqrt{a^2 + b^2} \text{ and}$$

$$PC = \sqrt{(a - 0)^2 + (b - 0)^2} = \sqrt{a^2 + b^2}.$$

3. Let the x-axis contain the segment and the y-axis contain its midpoint. Then the y-axis is the perpendicular bisector of the segment. Let $P(0,b)$ be any point of the y-axis, and $A(-a,0)$ and $B(a,0)$ be the end-points of the segment. Then:

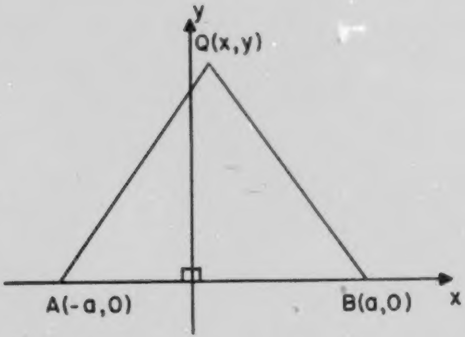


$$PA = \sqrt{(0 + a)^2 + (b - 0)^2} = \sqrt{a^2 + b^2}.$$

$$PB = \sqrt{(a - 0)^2 + (0 - b)^2} = \sqrt{a^2 + b^2}.$$

Hence $PA = PB$.

599 4. Place the axes so that the segment will have end-points $A(-a,0)$ $B(a,0)$, and the y-axis will be its perpendicular bisector. Let $Q(x,y)$ be any point equidistant from A and B. From the distance formula



$QA^2 = (x + a)^2 + y^2$ and $QB^2 = (x - a)^2 + y^2$.

Since $QA = QB$, $QA^2 = QB^2$ or

$$(x + a)^2 + y^2 = (x - a)^2 + y^2.$$

Simplifying, $4ax = 0$.

$$x = 0, \text{ since } a \neq 0.$$

Hence Q must lie on the y-axis which is the perpendicular bisector of $\overline{AB}$.

5. The mid-point of $\overline{AC} = (\frac{a + b}{2},\frac{c + 0}{2}) = (\frac{a + b}{2},\frac{c}{2})$.

The mid-point of $\overline{BD} = (\frac{a + b}{2},\frac{0 + c}{2}) = (\frac{a + b}{2},\frac{c}{2})$.

Since the diagonals have the same mid-points, they bisect each other.

6. $R = (\frac{d}{2},\frac{c}{2})$, $S = (\frac{b + a}{2},\frac{c}{2})$.

Since R and S have the same y-coordinates, $\overleftrightarrow{RS} \parallel \overleftrightarrow{AB}$.
Since $\overline{RS}$ is horizontal,

$$RS = \frac{b + a}{2} - \frac{d}{2} = \frac{b + a - d}{2}.$$

$$DC = d - b \text{ and } AB = a.$$

Therefore $\frac{1}{2}(AB - DC) = \frac{a - (d - b)}{2} = \frac{b + a - d}{2}$.

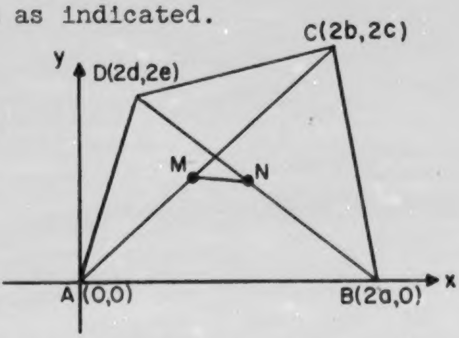
Hence, $RS = \frac{1}{2}(AB - DC)$ which was to be proved.

599 7. $R = (2a,0)$, $S = (2a + 2d, 2e)$.
$T = (2b + 2d, 2c + 2e)$, $W = (2b, 2c)$.
Mid-point of $\overline{WS} = (a + d + b, e + c)$.
Mid-point of $\overline{TR} = (a + b + d, c + e)$.
Therefore $\overline{WS}$ and $\overline{TR}$ bisect each other.

600 8. Area Δ ABC = area (XYBA) + area (YZCB) - area (XZCA).
Area $\Delta \text{ ABC} = \frac{1}{2}(s + r)(b - a) + \frac{1}{2}(t + s)(c - b) - \frac{1}{2}(r + t)(c - a)$.
Multiplying out and combining terms,
area $\Delta \text{ ABC} = \frac{1}{2}(rb - sa + sc - tb + ta - rc)$, or
area $\Delta \text{ ABC} = \dfrac{a(t - s) + b(r - t) + c(s - r)}{2}$

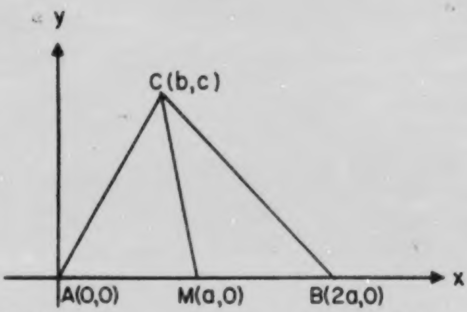
9. $ZY^2 = (b - a)^2 + c^2$. $XZ^2 = b^2 + c^2$.
$XY = a$. $XR = b$.
Since $(b - a)^2 + c^2 = (b^2 + c^2) + a^2 - 2ab$,
Therefore $ZY^2 = XZ^2 + XY^2 - 2XY \cdot XR$.
Observe that this proof remains valid if R lies between X and Y.

10. Select a coordinate system as indicated.



$M = (b,c)$, $N = (a + d, e)$.
$AB^2 = 4a^2$.
$BC^2 = 4(a - b)^2 + 4c^2$.
$CD^2 = 4(b - d)^2 + 4(c - e)^2$.
$DA^2 = 4d^2 + 4e^2$.
$AC^2 = 4b^2 + 4c^2$.
$BD^2 = 4(a - d)^2 + 4e^2$.
$MN^2 = (a + d - b)^2 + (e - c)^2$.
From these expressions the given equation can be verified. Note that
$(a + d - b)^2 = a^2 + d^2 + b^2 + 2ad - 2ab - 2bd$.

600 11. Place the axes and label the vertices as shown.


y
C(b,c)
A(0,0)
M(a,0)
B(2a,0)
x

$AC^2 = b^2 + c^2$.

$BC^2 = (2a - b)^2 + c^2$.

$\frac{AB^2}{2} = 2a^2$.

$MC^2 = (a - b)^2 + c^2$.

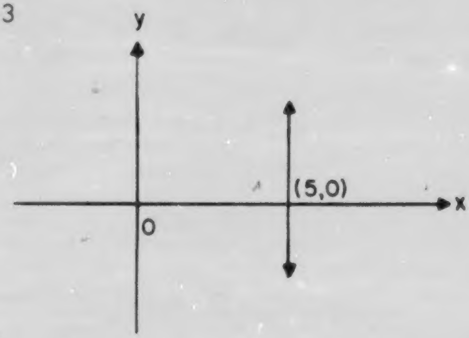
Since

$$(b^2 + c^2) + (4a^2 - 4ab + b^2 + c^2) = 2a^2 + 2(a^2 - 2ab + b^2 + c^2),$$
$$= 2a^2 + 2[(a - b)^2 + c^2].$$

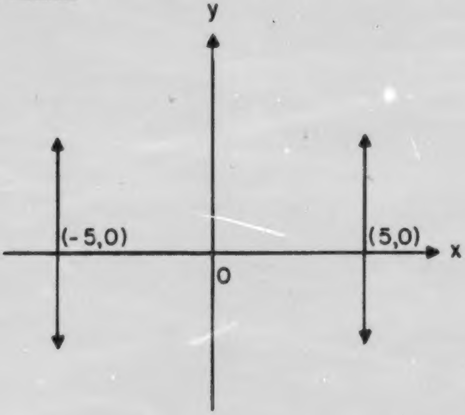
Therefore $AC^2 + BC^2 = \frac{AB^2}{2} + 2MC^2$.

Problem Set 17-9

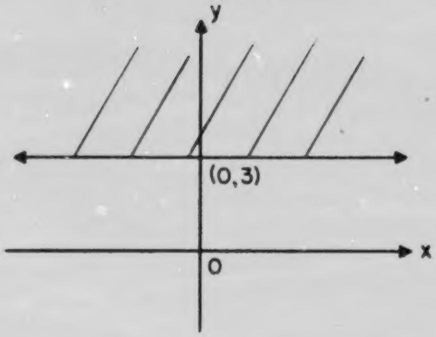
603



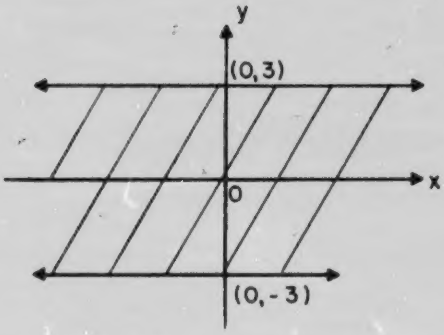
1a. The vertical line through (5,0).



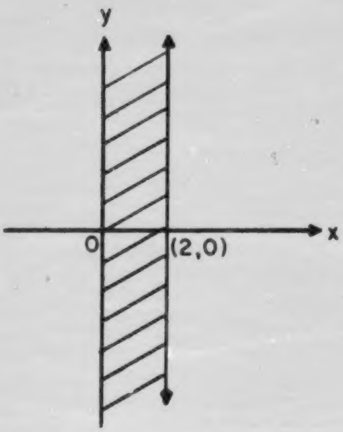
1b. The two vertical lines through (5,0) and (-5,0).



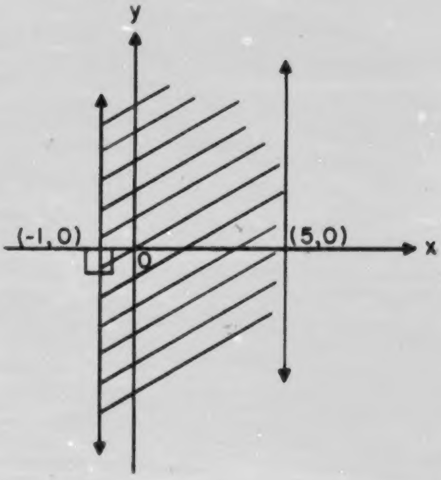
2a. The half-plane above the horizontal line through (0,3).



2b. All points between the lines $y = 3$ and $y = -3$.

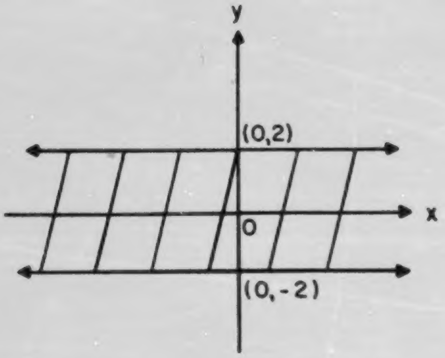


3. All points between the y-axis and the line $x = 2$.

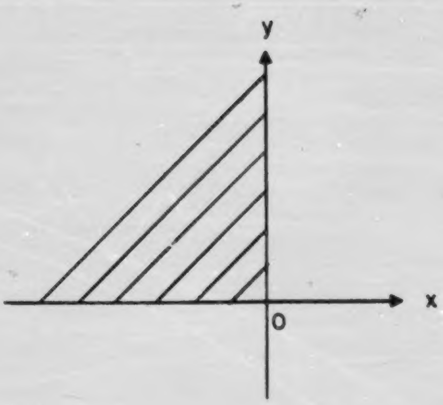


4. All points within or on the boundary of the indicated strip.

603

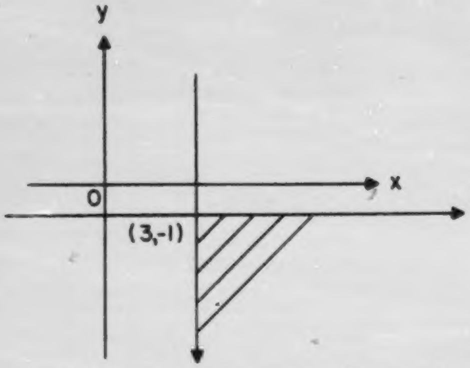


5. All points within, or on the lower boundary of the indicated strip.

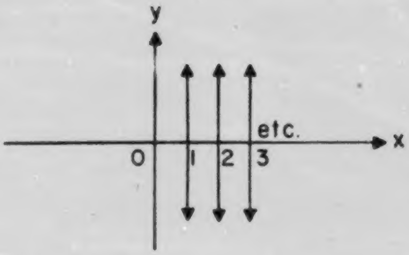


6. All points within the second quadrant.

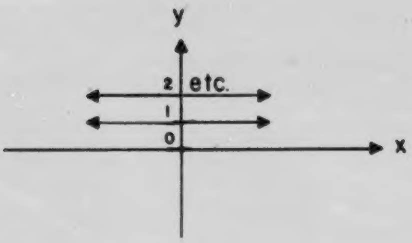
604



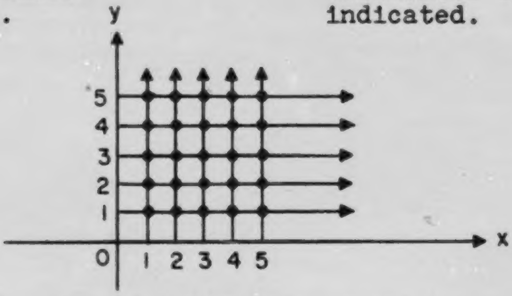
7. All points within indicated angle.



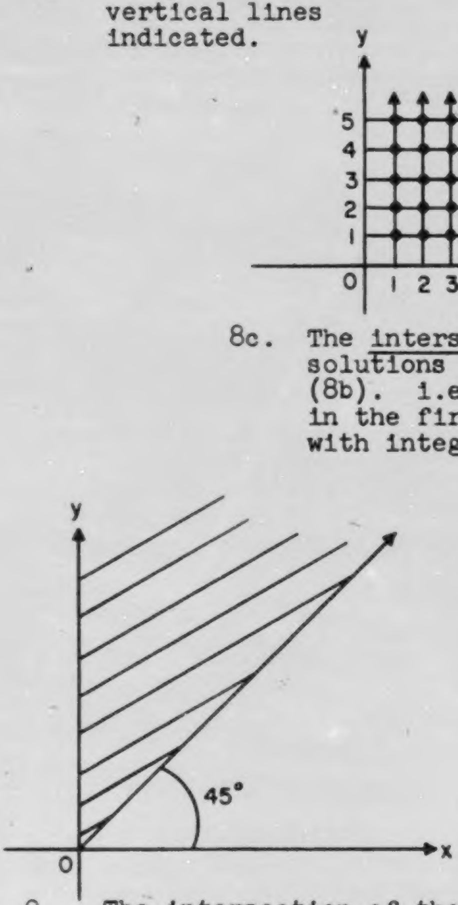
8a. All points on the vertical lines indicated.



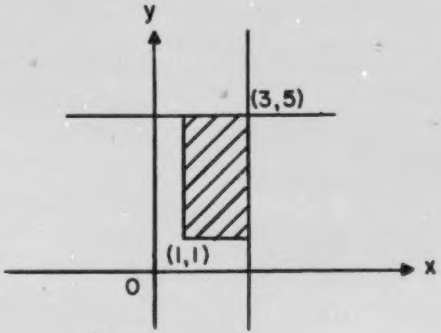
8b. All points on the horizontal lines indicated.



8c. The intersection of the solutions for (8a) and (8b). i.e., all points in the first quadrant with integral coordinates.

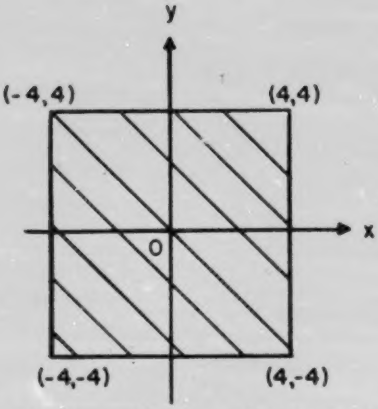


9. The intersection of the three half-planes formed by the three given conditions. i.e., all points within the angle formed by the positive part of the y-axis and the ray from the origin as shown.



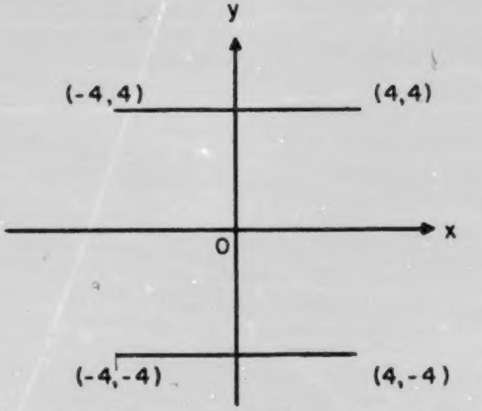
10. All points within or on the boundary of the indicated rectangle.

60^4 *11.



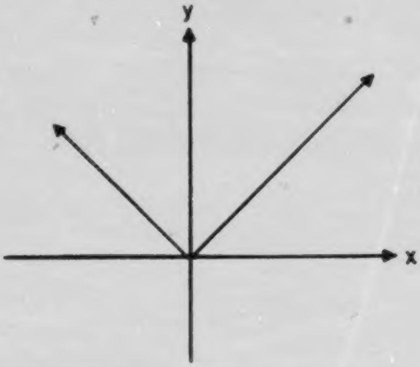
All points in the interior of the square with vertices (4,4), (-4,4), (-4,-4) (4,-4).

*12.



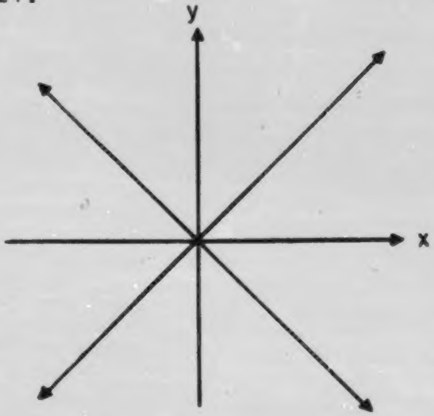
All points except the end-points on the two segments joining (-4,4) and (4,4), and (-4,-4) and (4,-4).

*13.



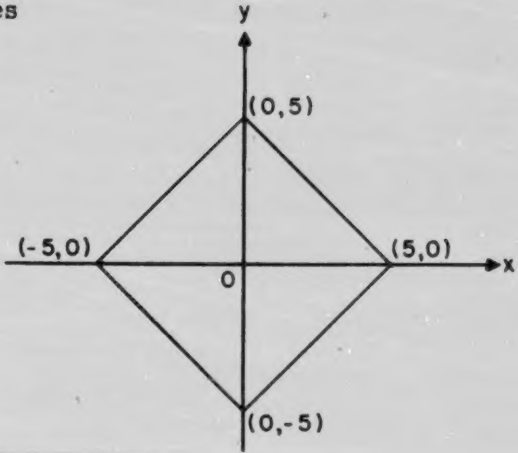
The rays bisecting the angles formed by the x and y-axes in first and second quadrants.

*14.



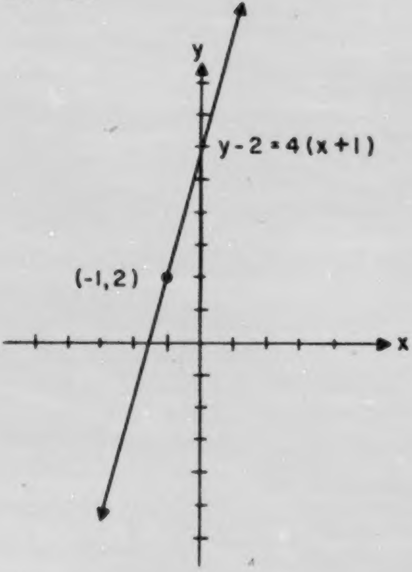
Lines bisecting the angles formed by the x and y-axes.

604 *15. The square with vertices (5,0), (0,5), (-5,0) and (0,-5).

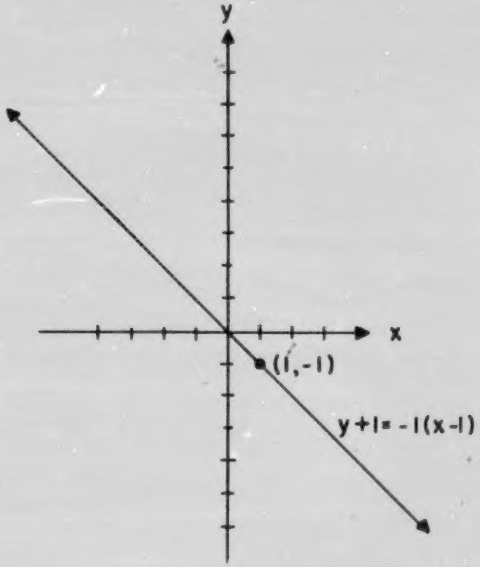

y
x
(0,5)
(-5,0)
(5,0)
0
(0,-5)

Problem Set 17-10

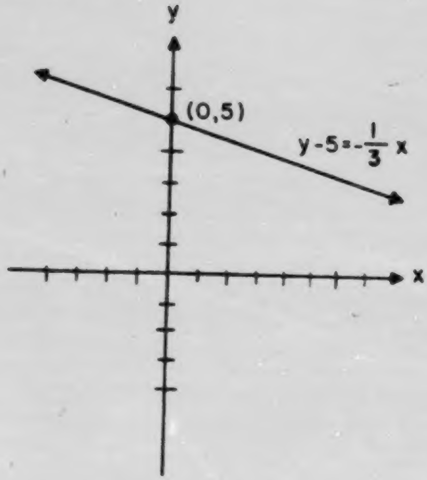
610 1.



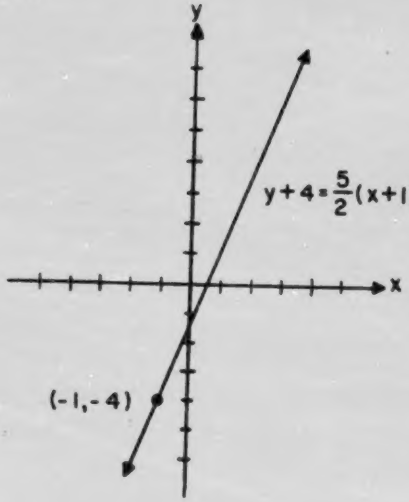
2.



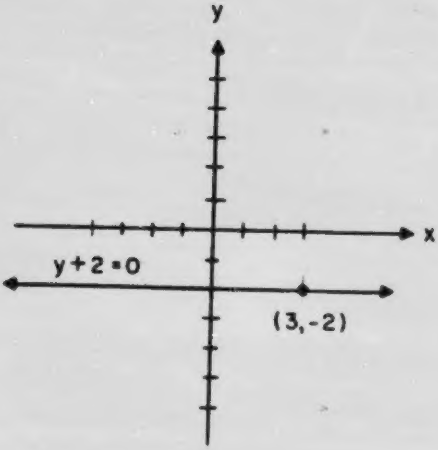
610 3.


y
(0,5)
$y-5=-\frac{1}{3}x$
x

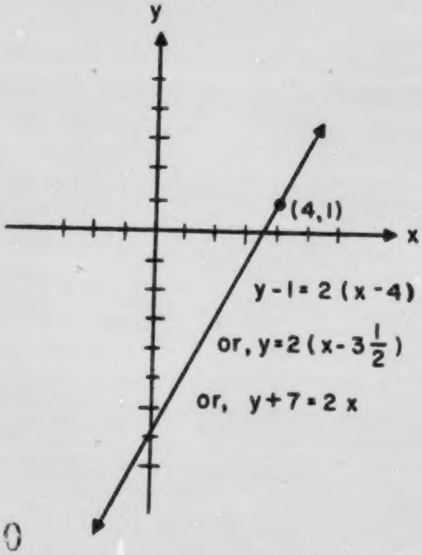
4.


y
$y+4=\frac{5}{2}(x+1)$
x
(-1,-4)

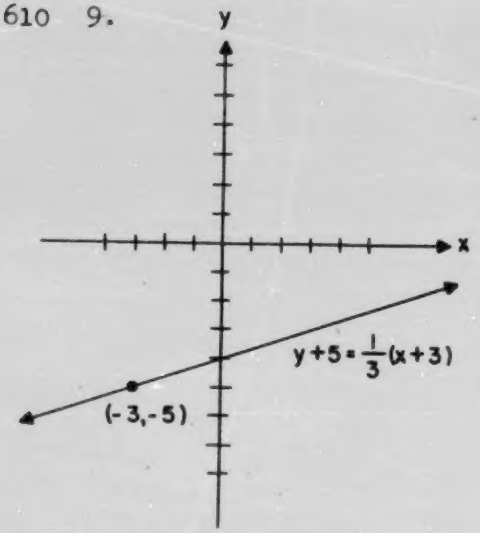
5.


y
x
$y+2=0$
(3,-2)

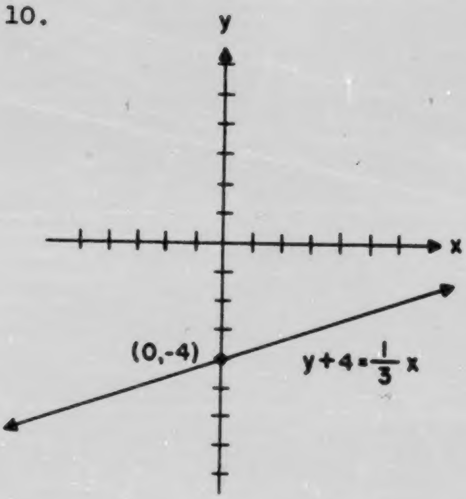
6,7,8.


y
(4,1)
x
$y-1=2(x-4)$
or, $y=2(x-3\frac{1}{2})$
or, $y+7=2x$

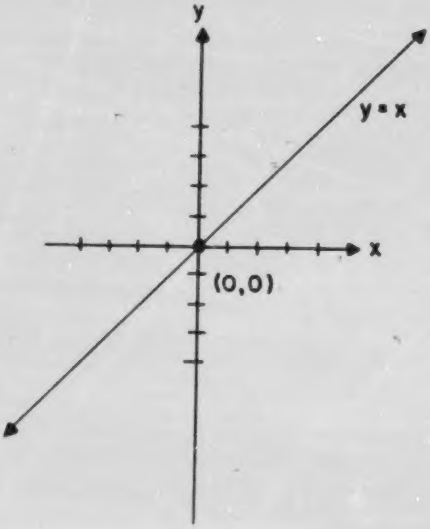
610 9.



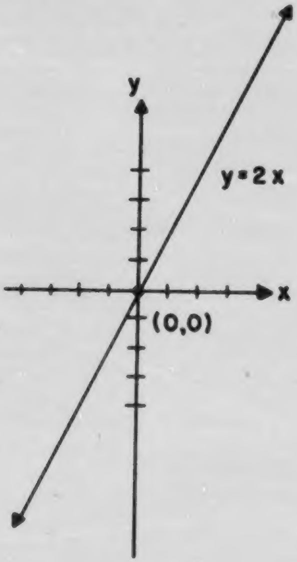
10.



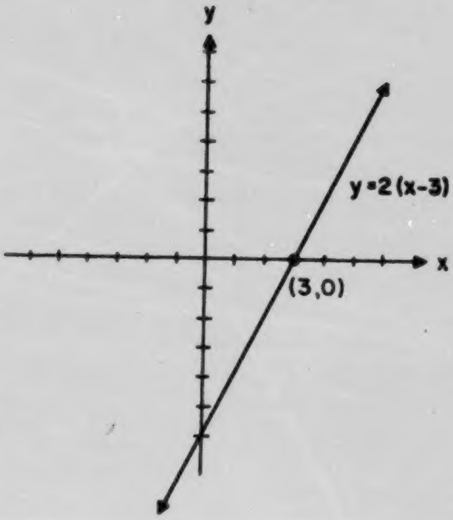
11.



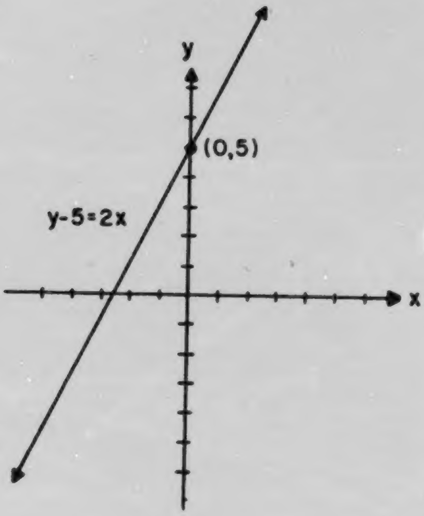
12.



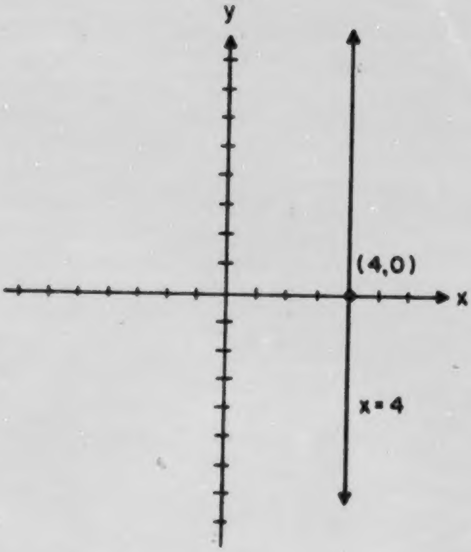
610 13.



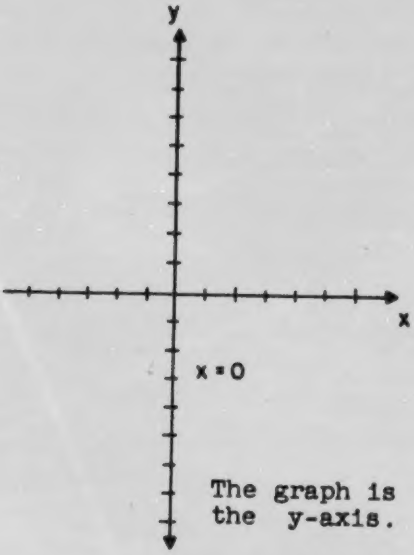
14.



15.



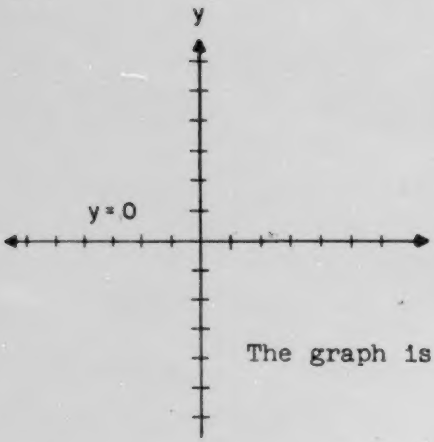
16.



The graph is the y-axis.

[page 610]

610 17.



The graph is the x-axis.

611 18. a. The yz-plane.

b. The xy-plane.

c. A plane parallel to the yz-plane, intersecting the x-axis at $x = 1$.

d. A plane parallel to the xz-plane, intersecting the y-axis at $y = 2$.

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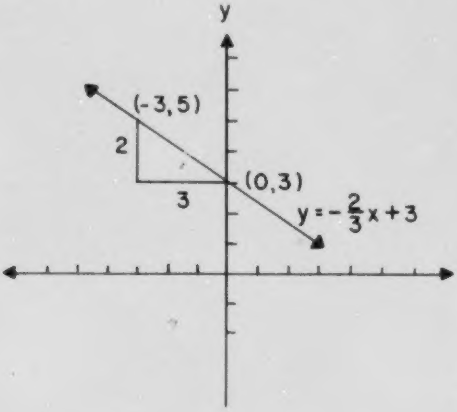
611 The material in Section 17-11 may have been previously covered in a first year algebra course. If this is the case, do not spend any more time than is necessary on this section.

You will note that in the discussion on this page, it is necessary for us to find an additional point in order to plot the graph of the equation. We may do this in two ways. The first would be to assign to x a value, substitute this value in the given equation and compute the corresponding value of y (or we could assign a value to y and compute x). The second method depends upon the discussion here in the text. For we know how a line with a positive or negative slope will lie, and we also know that if a line has a positive

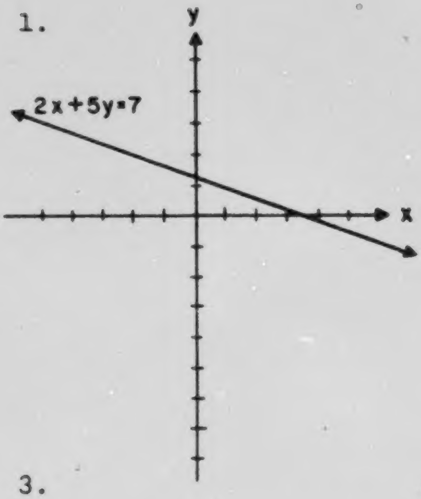
611 slope then $m = \frac{RP_2}{P_1R}$ and if its slope is negative, $m = -\frac{RP_2}{P_1R}$. Then, given one point on the graph and the slope we can find a second point by counting the units in the legs of the right triangle. Consider the example used by the text, $y = 3x - 4$. We see immediately that the y-intercept is -4 and that the slope is 3. Since the slope is positive, the graph will rise to the right. Hence, we can find a second point by starting at $(0,-4)$ and counting 1 unit to the right and three units up to the point $(1,-1)$. We can check to see that we are correct by applying the slope formula to these coordinates.

Let us consider one more case, namely, when the slope of the given line is negative. Draw the graph of the equation $y = -\frac{2}{3}x + 3$. We see that the point $(0,3)$ lies on the graph, and to locate a second point by this method, we must realize that we will be working with a slope of $-\frac{2}{3}$. The graph, then, will rise to the left and we can locate a second point by counting 3 units to the left from $(0,3)$ and 2 units up, as in the figure below.

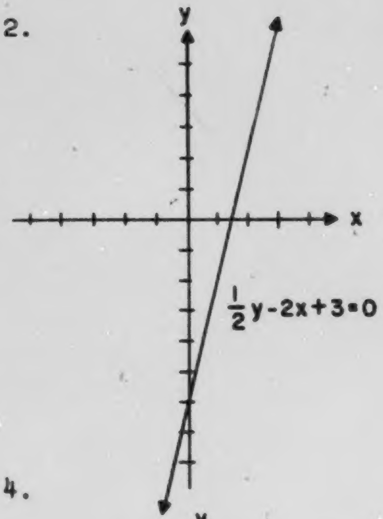


Problem Set 17-12

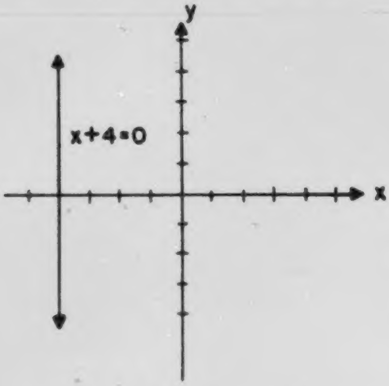
616 1.



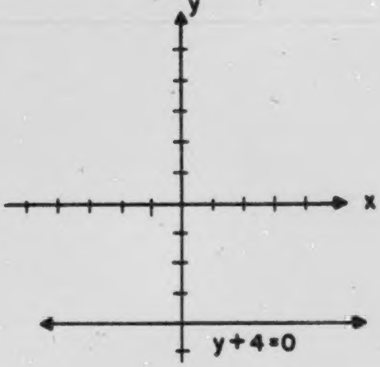
2.



3.



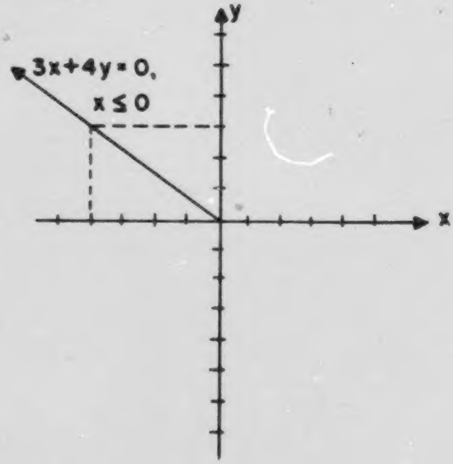
4.



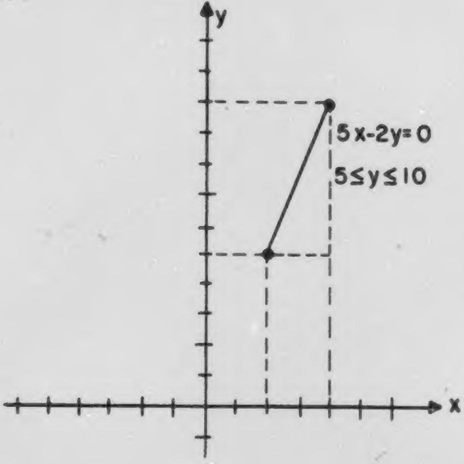
5. The graph is the whole xy-plane.
6. The graph is the empty set; i.e., there are no points whose coordinates satisfy the equation.
7. The graph contains a single point, the origin (0,0).
8. The graph is the empty set.

[page 616]

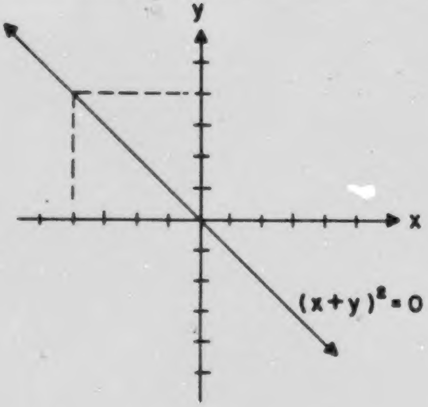
9.



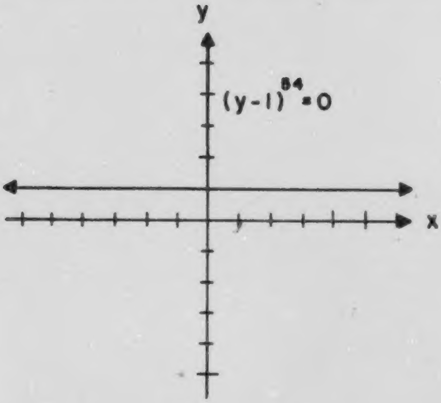
10.



11.



12.



13. $3x - y - 1 = 0$. $A = 3$, $B = -1$, $C = -1$.

14. $x + y - 1 = 0$. $A = 1$, $B = 1$, $C = -1$.

15. $2x - y - 4 = 0$. $A = 2$, $B = -1$, $C = -4$.

16. $y = 0$. $A = 0$, $B = 1$, $C = 0$.

17. $x = 0$. $A = 1$, $B = 0$, $C = 0$.

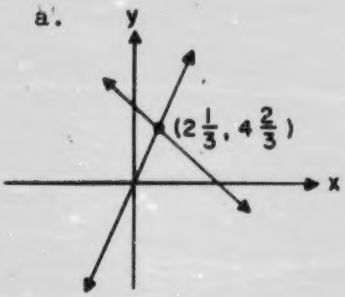
18. $y + 3 = 0$. $A = 0$, $B = 1$, $C = 3$.

19. $x + 5 = 0$. $A = 1$, $B = 0$, $C = 5$.

20. $x - 5y = 0$. $A = 1$, $B = -5$, $C = 0$.

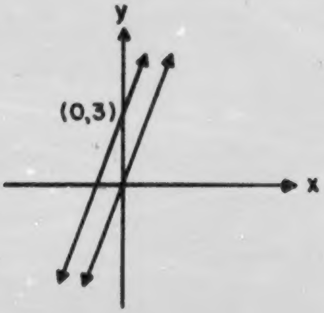
Problem Set 17-13

618 1. a.


y
x
$(2\frac{1}{3}, 4\frac{2}{3})$

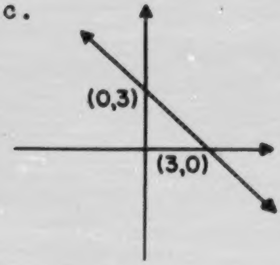
$x = 2\frac{1}{3}$; $y = 4\frac{2}{3}$.

b.



The empty set.

c.



The equations are
equivalent. Any
pair of numbers
whose sum is 3
is a common solution.

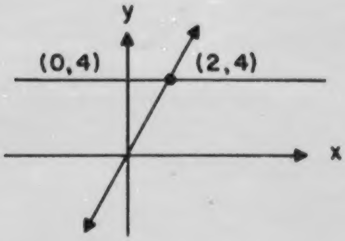
2. a. (1) and (4), (3) and (4).

   b. (1) and (2), (2) and (3), (2) and (4).

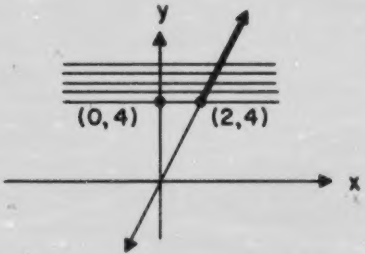
   c. (1) and (3).

3. 4000 miles.

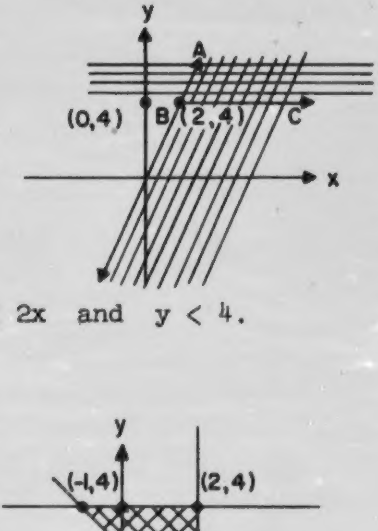
619 4. a. The intersection is point $(2,4)$.



b. The intersection is the ray shown with end-point $(2,4)$.



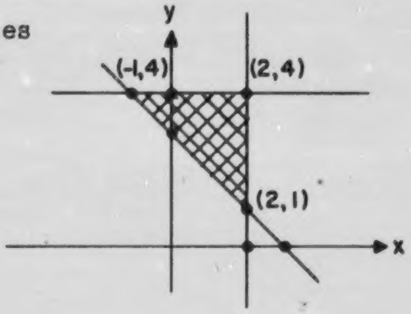
c. The intersection is the interior of $\angle ABC$.



d. The conditions are $y < 2x$ and $y < 4$.

5. a. The intersection is the interior of the triangle with vertices $(2,1)$, $(2,4)$, and $(-1,4)$.

b. $x + y < 3$,
$x > 0$,
$y > 0$.



619 6. The mid-point M has coordinates

$(\frac{3+5}{2}, \frac{4+8}{2}) = (4,6)$.

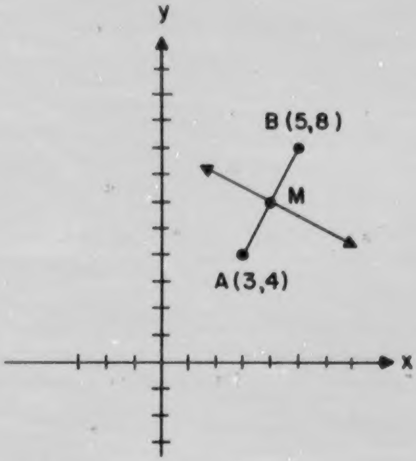
The slope of $\overline{AB}$ is

$\frac{8-4}{5-3} = 2$, so the slope of L is $-\frac{1}{2}$ and its equation is L:

$y - 6 = -\frac{1}{2}(x - 4)$,

$y - 6 = -\frac{1}{2}x + 2$, or

$x + 2y = 16$.



Alternate solution: L is the set of points P(x,y) for which PA = PB. This gives

$$\sqrt{(x-3)^2 + (y-4)^2} = \sqrt{(x-5)^2 + (y-8)^2}$$

which reduces to $x + 2y = 16$.

7. In the preceding problem, we found the equation

L: $x + 2y = 16$.

Similarly, for M and N we find

M: $3x - y = -3$,

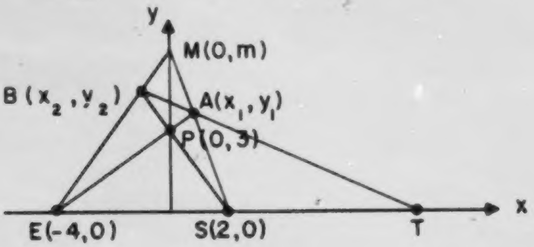
N: $2x - 3y = -19$.

The intersection G of L and M is obtained by solving their equations:

$G = (\frac{10}{7}, \frac{51}{7})$.

Substituting in the third equation, we find that G lies on N also.

620 *8. Take a coordinate system in which Queen's Road is the x-axis and King's Road is the y-axis.



The coordinates of the elm, spruce, and pine are as indicated. The maple is gone, but its assumed position is labeled (0,m). The slope of $\overleftrightarrow{EP}$ is $\frac{3}{4}$, so its equation (in slope-intercept form) is

$$\overleftrightarrow{EP}: \quad y = \tfrac{3}{4}x + 3.$$

The slope of $\overleftrightarrow{SM}$ is $-\frac{m}{2}$, so its equation (in point-slope form) is

$$\overleftrightarrow{SM}: \quad y = -\tfrac{m}{2}(x - 2).$$

Solving these two equations simultaneously, we find the coordinates of A:

$$A: \begin{cases} x_1 = \dfrac{4(m - 3)}{2m + 3}, \\[2ex] y_1 = \dfrac{9m}{2m + 3}. \end{cases}$$

Similarly, we get the equations

$$\overleftrightarrow{SP}: \quad y = -\tfrac{3}{2}x + 3,$$

$$\overleftrightarrow{EM}: \quad y = \tfrac{m}{4}(x + 4),$$

and the point of intersection is

$$B: \begin{cases} x_2 = -\dfrac{4(m - 3)}{m + 6}, \\[2ex] y_2 = \dfrac{9m}{m + 6}. \end{cases}$$

The line $\overleftrightarrow{AB}$ has the equation,

$$\overleftrightarrow{AB}: \quad y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1).$$

620 The intersection T of $\overleftrightarrow{AB}$ and the x-axis is found by letting $y = 0$ and solving for x:

$$x = x_1 - y_1\left(\frac{x_2 - x_1}{y_2 - y_1}\right),$$

$$x = \frac{x_1y_2 - x_2y_1}{y_2 - y_1}.$$

Now

$$x_1y_2 - x_2y_1 = \frac{4(m - 3)}{2m + 3} \cdot \frac{9m}{m + 6} + \frac{4(m - 3)}{m + 6} \cdot \frac{9m}{2m + 3}.$$

$$= \frac{72m(m - 3)}{(m + 6)(2m + 3)},$$

$$y_2 - y_1 = \frac{9m}{m + 6} - \frac{9m}{2m + 3},$$

$$= \frac{9m(m - 3)}{(m + 6)(2m + 3)}.$$

Dividing, we get $x = 8$. Therefore the treasure was buried 8 miles east of the crossing.

Suppose now that the pine were also missing. Assume coordinates $(0,p)$, for P and carry through the calculation in terms of both m and p. The algebra is a little more complicated, but if it is done correctly both m and p drop out in the final result, which is again $x = 8$.

620 *9. The y-axis is a line through C, perpendicular to the base $\overline{AB}$, i.e., it contains the altitude from C. If $\overleftrightarrow{AM}$ where m is its slope, contains the altitude from A, it has the equation

$$y = m(x + 4),$$

Since $\overleftrightarrow{AM} \perp \overleftrightarrow{BC}$, $m = -\dfrac{1}{\text{slope } \overleftrightarrow{BC}}$.

But slope $\overleftrightarrow{BC} = -\frac{8}{7}$, so $m = \frac{7}{8}$, and the equation of $\overleftrightarrow{AM}$ is $y = \frac{7}{8}(x + 4)$.

To find the y-intercept, let $x = 0$:

$$y = \frac{7}{8} \cdot 4 = \frac{7}{2}.$$

Now do the same for $\overleftrightarrow{BN}$, which contains the altitude from B. Slope $\overleftrightarrow{AC} = \frac{8}{4} = 2$, so the slope of $\overleftrightarrow{BN}$ is $-\frac{1}{2}$, and its equation is

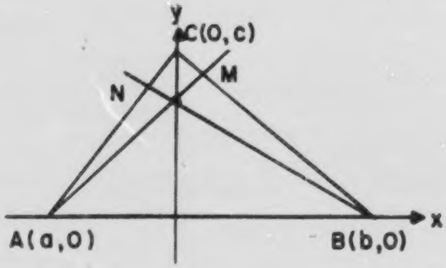
$$y = -\frac{1}{2}(x - 7).$$

Letting $x = 0$, we get the y-intercept

$$y = -\frac{1}{2}(-7) = \frac{7}{2}.$$

Therefore $\overleftrightarrow{AM}$ and $\overleftrightarrow{BN}$ meet at the point $(0,\frac{7}{2})$ on the line containing the altitude from C.

For the general triangle,



slope $\overleftrightarrow{BC} = -\frac{c}{b}$,

slope $\overleftrightarrow{AM} = \frac{b}{c}$, so

$\overleftrightarrow{AM}$: $y = \frac{b}{c}(x - a)$, and

the y-intercept is $-\frac{ba}{c}$.

Similarly, slope $\overleftrightarrow{AC} = -\frac{c}{a}$,

slope $\overleftrightarrow{BN} = \frac{a}{c}$, so

$\overleftrightarrow{BN}$: $y = \frac{a}{c}(x - b)$, and

the y-intercept is $-\frac{ab}{c}$.

620 Therefore the three altitudes meet at the point $(0, -\frac{ab}{c})$. Note that this proof does not depend on the signs of a, b, and c, but only on the fact that A, B, lie on the x-axis and C on the y-axis.

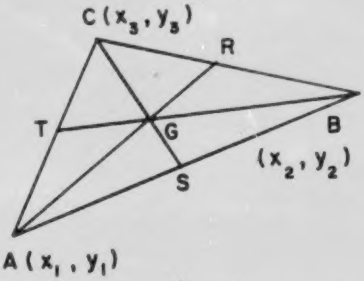
621 *10. Let $A = (x_1, y_1)$, $B = (x_2, y_2)$, $C = (x_3, y_3)$.

Then we have

$$R = (\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}),$$

$$S = (\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}),$$

$$T = (\frac{x_1 + x_3}{2}, \frac{y_1 + y_3}{2}).$$



The slope of $\overleftrightarrow{AR}$ is

$$m_1 = \frac{\frac{y_2 + y_3}{2} - y_1}{\frac{x_2 + x_3}{2} - x_1} = \frac{y_2 + y_3 - 2y_1}{x_2 + x_3 - 2x_1}.$$

If $G = (\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3})$, then the slope of $\overleftrightarrow{AG}$ is

$$m_1' = \frac{\frac{y_1 + y_2 + y_3}{3} - y_1}{\frac{x_1 + x_2 + x_3}{3} - x_1} = m_1,$$

so G is on the median $\overleftrightarrow{AR}$. Similarly, the slope of $\overleftrightarrow{BT}$ is

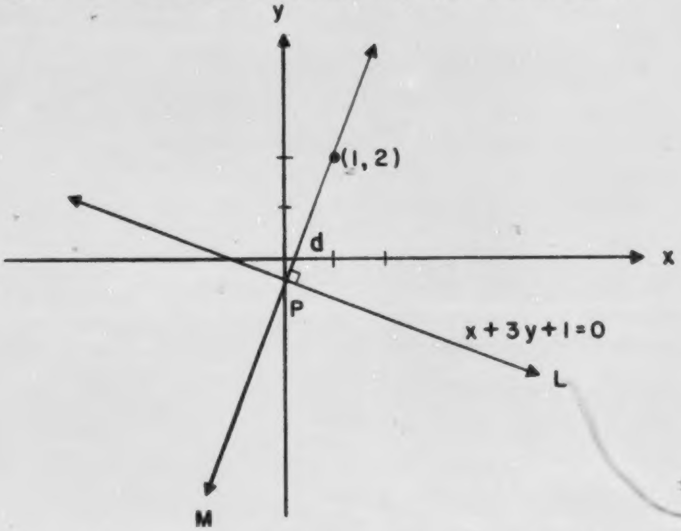
$$m_2 = \frac{\frac{y_1 + y_3}{2} - y_2}{\frac{x_1 + x_3}{2} - x_2} = \frac{y_1 + y_3 - 2y_2}{x_1 + x_3 - 2x_2},$$

621 and the slope of $\overleftrightarrow{BG}$ is

$$m_2' = \frac{\dfrac{y_1 + y_2 + y_3}{3} - y_2}{\dfrac{x_1 + x_2 + x_3}{3} - x_2} = m_2,$$

so G is on the median $\overleftrightarrow{BG}$. Similarly, we find that G is on the median $\overleftrightarrow{CS}$. Hence, the three medians intersect in the point G whose coordinates are the averages of the coordinates of the vertices.

*11.



The equation $x + 3y + 1 = 0$ is equivalent to $y = -\frac{1}{3}x - \frac{1}{3}$, which is in slope-intercept form. Therefore the slope is $-\frac{1}{3}$. The line M through (1,2) perpendicular to L has slope 3, so an equation for it is

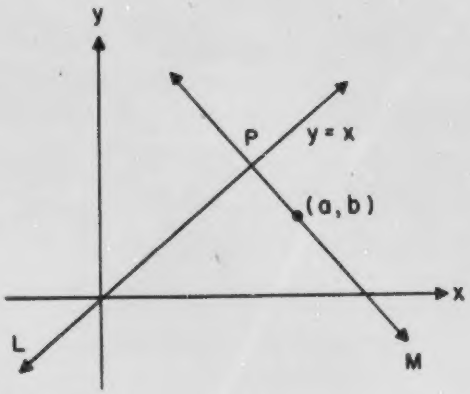
$$\begin{aligned} M:\quad y - 2 &= 3(x - 1), \\ y &= 3x - 1. \end{aligned}$$

Solving the equations for M and L simultaneously to find their intersection P, we get

$$P = \left(\frac{1}{5}, -\frac{2}{5}\right).$$

Computing the distance d from (1,2) to P by the distance formula, we find $d = \frac{4}{5}\sqrt{10}$.

621 *12. The line L with equation $y = x$ has slope 1, so the line M through (a,b) perpendicular to L has slope -1. An equation for M is



$$M: \quad y - b = -(x - a),$$
$$x + y = a + b.$$

Solving for the point of intersection P, we get

$$P = \left(\frac{a + b}{2}, \frac{a + b}{2}\right).$$

The distance is obtained from $d^2 = \left(\frac{a + b}{2} - a\right)^2 + \left(\frac{a + b}{2} - b\right)^2 = \frac{(a - b)^2}{2}$,

$$d = \frac{|a - b|}{\sqrt{2}}.$$

*13. From Problem 9, we have $H = \left(0, -\frac{ab}{c}\right)$.

From Problem 10, we have $M = \left(\frac{a + b}{3}, \frac{c}{3}\right)$.

To find D we get the perpendicular bisectors u, v of $\overline{AB}$ and $\overline{BC}$:

$$u: \quad x = \frac{a + b}{2},$$

$$v: \quad y - \frac{c}{2} = \frac{b}{c}\left(x - \frac{b}{2}\right).$$

Therefore, $\quad D = \left(\frac{a + b}{2}, \frac{c^2 + ab}{2c}\right).$

Now

$$HM^2 = \left(\frac{a + b}{3}\right)^2 + \left(\frac{c^2 + 3ab}{3c}\right)^2 = \frac{c^2(a + b)^2 + (c^2 + 3ab)^2}{(3c)^2},$$

$$HD^2 = \left(\frac{a + b}{2}\right)^2 + \left(\frac{c^2 + 3ab}{2c}\right)^2 = \frac{c^2(a + b)^2 + (c^2 + 3ab)^2}{(2c)^2},$$

$$MD^2 = \left(\frac{a + b}{6}\right)^2 + \left(\frac{c^2 + 3ab}{6c}\right)^2 = \frac{c^2(a + b)^2 + (c^2 + 3ab)^2}{(6c)^2}.$$

621 From these equations we get,

$$HM = 2MD, \quad HD = 3MD,$$

$$HM + MD = HD.$$

This shows that H, M, and D are collinear, that M is between H and D, and that M trisects $\overline{HD}$: $MD = \frac{1}{3} HD$.

Problem Set 17-14

626 1. In each case the result is 25. This becomes obvious if radii are drawn to the points on the circle.

2. a. (1), (3), (4), (6).
   b. (3), (4).
   c. (1).

3. a. Center (0,0); $r = 3$.
   b. (0,0); $r = 10$.
   c. (1,0); $r = 4$.
   d. (0,0); $r = \sqrt{7}$.
   e. (0,0); $r = 2$.
   f. (4,3); $r = 6$.
   g. (-1,-5); $r = 7$.
   h. (1,0); $r = 5$.
   i. (1,0); $r = 5$.
   j. (-3,2); $r = 5$.

627 4. a. Replacing x and y in the equation by the given coordinates satisfies the equation.

   b. $x^2 - 10x + y^2 = 0,$

   $(x^2 - 10x + 25) + y^2 = 25,$

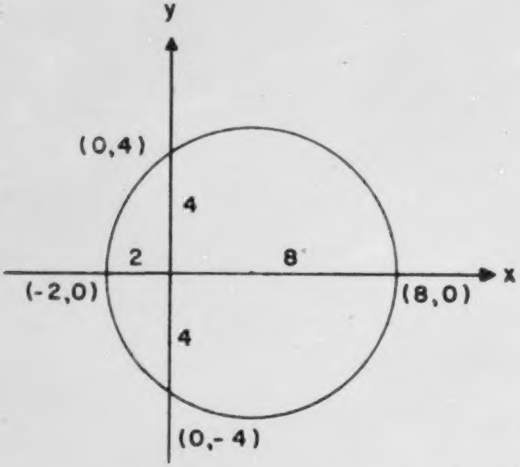
   $(x - 5)^2 + (y - 0)^2 = 5^2.$

   The center of the circle is (5,0); the radius is 5.

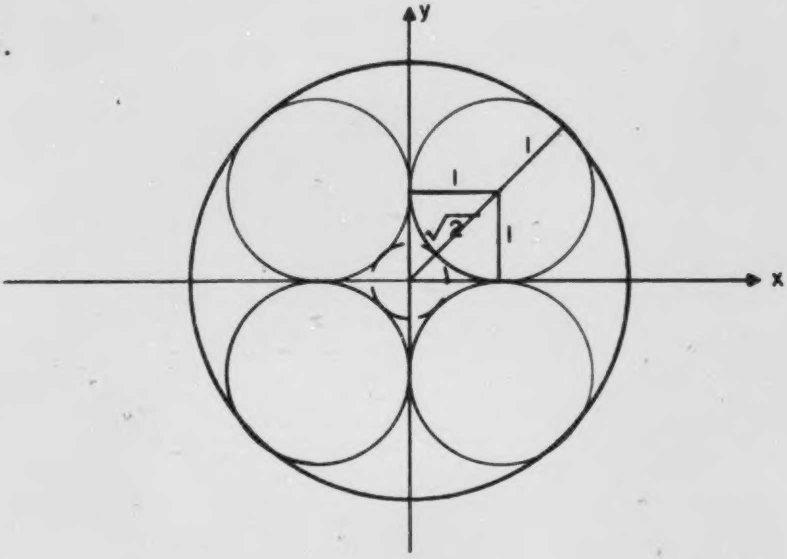
627 c. The ends of the diameter along the x-axis are $(0,0)$ and $(10,0)$. The slope of the segment joining $(0,0)$ and $(1,3)$ is 3. The slope of the segment joining $(10,0)$ and $(1,3)$ is $-\frac{1}{3}$. Since 3 and $-\frac{1}{3}$ are negative reciprocals, the lines are perpendicular and a right angle is formed.

5. a. The x-axis intersects the circle where $y = 0$, that is where $(x - 3)^2 = 25$, or at points $(-2,0)$ and $(8,0)$. The y-axis intersects the circle where $x = 0$, that is where $9 + y^2 = 25$, or at points $(0,4)$ and $(0,-4)$.

b. $2 \cdot 8 = 4 \cdot 4 = 16$.



627 6.

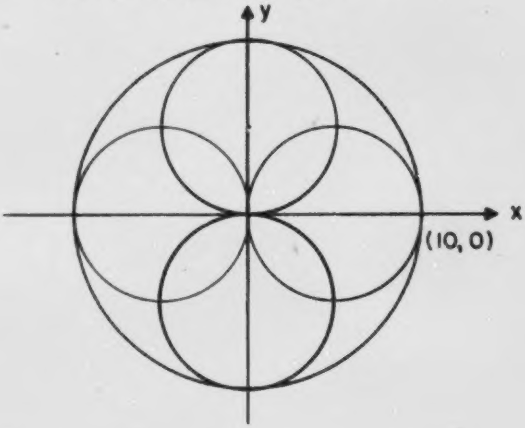


The radius of the larger circle is $1 + \sqrt{2}$. So the equation is

$$x^2 + y^2 = (1 + \sqrt{2})^2.$$

There would be another tangent circle of radius $\sqrt{2} - 1$ and the same center.

7.



The including circle is $x^2 + y^2 = 100$.

627 8. a. $y = m(x + 7)$.

b. $x^2 + [m(x + 7)]^2 = 16$,

$$(1 + m^2)x^2 + 14m^2x + (49m^2 - 16) = 0,$$

$$x = \frac{-14m^2 \pm \sqrt{(14m^2)^2 - 4(1 + m^2)(49m^2 - 16)}}{2(1 + m^2)}$$

$$= \frac{-14m^2 \pm \sqrt{4(16 - 33m^2)}}{2(1 + m^2)}$$

$$= \frac{-7m^2 \pm \sqrt{16 - 33m^2}}{1 + m^2}$$

$$y = m(x + 7) = \left(\frac{-7m^2 \pm \sqrt{16 - 33m^2} + 7 + 7m^2}{1 + m^2}\right)m$$

$$= \frac{m(7 \pm \sqrt{16 - 33m^2})}{1 + m^2}.$$

If $16 - 33m^2 > 0$, there are two points of intersection:

$$P_1 = \left(\frac{-7m^2 + \sqrt{16 - 33m^2}}{1 + m^2}, \frac{m(7 + \sqrt{16 - 33m^2})}{1 + m^2}\right),$$

$$P_2 = \left(\frac{-7m^2 - \sqrt{16 - 33m^2}}{1 + m^2}, \frac{m(7 - \sqrt{16 - 33m^2})}{1 + m^2}\right).$$

627 c. If $16 - 33m^2 = 0$, there is one point of intersection:

$$P = \left(\frac{-7m^2}{1 + m^2}, \frac{7m}{1 + m^2}\right)$$

and $m^2 = \frac{16}{33}$, $m = \pm \frac{4}{\sqrt{33}}$.

This means that the two lines

$$y = \frac{4}{\sqrt{33}}(x + 7),$$

$$y = -\frac{4}{\sqrt{33}}(x + 7)$$

are tangent to the circle.

If $16 - 33m^2 < 0$, there is no point of intersection.

628 9. Put the given equation in standard form

$$(x - 5)^2 + (y - 3)^2 = 2^2.$$

The given circle has center $(5,3)$, radius 2. Let the required circle have center (a,b) and radius r. Then $b = a = r$, since the circle touches the x- and y-axes, and the distance from center (a,b) to center $(5,3)$ is $r + 2$. Hence,

$$r + 2 = \sqrt{(r - 5)^2 + (r - 3)^2}$$

$$r^2 + 4r + 4 = 2r^2 - 16r + 34$$

$$r^2 - 20r + 30 = 0$$

$$r = \frac{20 \pm \sqrt{400 - 120}}{2}$$

$$r = 10 \pm \sqrt{70}.$$

Thus, there are two solutions:

$$(x - r)^2 + (y - r)^2 = r^2,$$

where $r = 10 + \sqrt{70}$ (approx. 18.37) and $r^2 = 337.3$ (approx.) or $10 - \sqrt{70}$ (approx. 1.63) and $r^2 = 2.7$ (approx.).

Review Problems

628 1. (5,0).

2. (-1,5).

3. $\frac{b}{3a}$. The median is vertical and has no slope.

4. $\frac{a}{b}$.

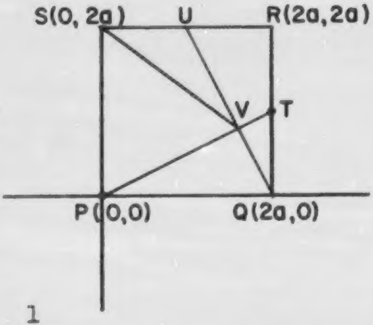
5. 2b; $\sqrt{9a^2 + b^2}$; $\sqrt{9a^2 + b^2}$.

6. $-\frac{3}{2}$.

7. $5\sqrt{2}$; $6\sqrt{2}$.

8. $(\frac{1}{2},3\frac{1}{2})$; (3,6); (6,3); $(3\frac{1}{2},\frac{1}{2})$; (4,4); $(2\frac{1}{2},2\frac{1}{2})$.

9. Place the axes and assign coordinates as shown.



a. T = (2a,a), U = (a,2a).

$PT = \sqrt{4a^2 + a^2} = a\sqrt{5}$.

$QU = \sqrt{a^2 + 4a^2} = a\sqrt{5}$.

Therefore PT = QU.

b. The slope of $\overline{PT} = \frac{a - 0}{2a - 0} = \frac{1}{2}$.

The slope of $\overline{QU} = \frac{0 - 2a}{2a - a} = -2$.

Since -2 is the negative reciprocal of $\frac{1}{2}$, the segments are perpendicular.

628 *c. Using the point-slope form the equation of $\overleftrightarrow{PT}$ is:

$$y - 0 = \frac{1}{2}(x - 0)$$

or $$y = \frac{1}{2}x.$$

The equation of $\overleftrightarrow{QU}$ is:

$$y - 0 = -2(x - 2a)$$

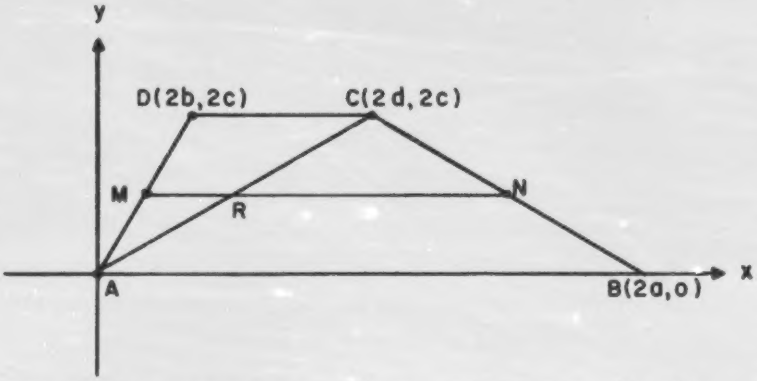
or $$y = -2x + 4a.$$

The coordinates of V, given by the common solution of the equations of $\overleftrightarrow{PT}$ and $\overleftrightarrow{QU}$ are $(\frac{8a}{5}, \frac{4a}{5})$. The distance VS is then

$$\sqrt{(\frac{8a}{5} - 0)^2 + (\frac{4a}{5} - 2a)^2} = \sqrt{\frac{100a^2}{25}} = 2a = \text{length}$$

of side.

629 10.



Take coordinate system as shown. Then $M = (b,c)$; $N = (a + d,c)$.

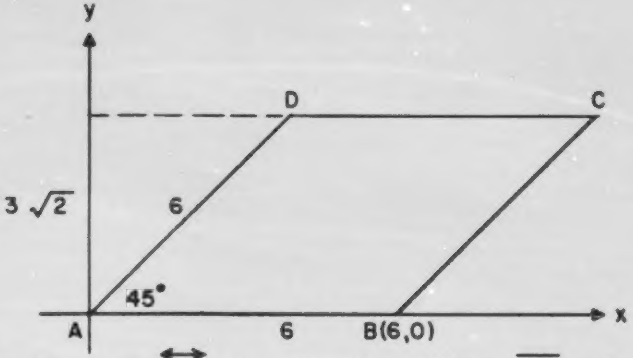
Equation of $\overleftrightarrow{MN}$ is: $y = c$.

Equation of diagonal $\overline{AC}$ is: $y = \frac{c}{d}x$.

Point R of intersection is (d,c), which is also the mid-point of $\overline{AC}$.

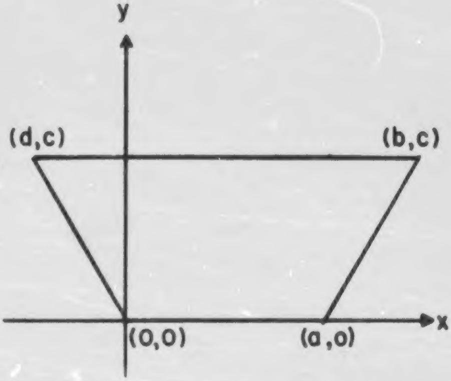
11. $x = 0$.

629 12.



Equation of $\overleftrightarrow{AB}$ is $y = 0$. Slope $\overline{BC} = 1$.
Equation of $\overleftrightarrow{BC}$ is $y = x - 6$.
Equation of $\overleftrightarrow{CD}$ is $y = 3\sqrt{2}$.

13. Lengths of parallel sides are: $|a|$, $|b - d|$.
Altitude is $|c|$.
Hence,
area $= \frac{1}{2}|c|(|a| + |b - d|)$.



14. (2,1).

15. A circle with center at the origin and radius 2.

16. a. $x^2 + y^2 = 49$.

b. $x^2 + y^2 = k^2$.

c. $(x - 1)^2 + (y - 2)^2 = 9$.

*17. Find first the intersection of the line $x + y = 2$ and the circle. Now $x = 2 - y$.

Therefore,
$$(2 - y)^2 + y^2 = 2,$$
$$4 - 4y + y^2 + y^2 = 2,$$
$$(y - 1)^2 = 0,$$
so that $y = 1$ and $x = 1$.

Thus the point (1,1) is the only point of intersection, so that the line is tangent to the circle.

[page 629]

Answers to Review Exercises

Chapters 13 to 17

1. 1.
2. 1.
3. 0.
4. 1.
5. 0.
6. 0.
7. 0.
8. 1.
9. 1.
10. 0.
11. 0.
12. 0.
13. 0.
14. 1.
15. 0.
16. 1.
17. 1.
18. 0.
19. 1.
20. 1.
21. 0.
22. 0.
23. 1.
24. 1.
25. 0.
26. 1.
27. 0.
28. 1.
29. 1.
30. 0.
31. 0.
32. 1.
33. 1.
34. 1.
35. 1.
36. 1.
37. 0.
38. 0.
39. 1.
40. 0.
41. 1.
42. 0.
43. 0.
44. 0.
45. 1.
46. 0.
47. 1.
48. 0.
49. 1.
50. 0.

[pages 630-633]

Illustrative Test Items for Chapter 17

A. 1. What name is given to the projection of the point (5,0) into the y-axis.

2. State the number of the quadrant in which each of the following points is located: (3,3), (6,-2), (-2,8).

3. What are the coordinates of a point on the x-axis if the distance from the point to the y-axis is 4?

4. A ray with its end-point at the origin makes a 30° angle with the positive x-axis and extends into the first quadrant. What are the coordinates of a point on the ray whose distance from the origin is 2?

B. 1. Determine the slopes of the line segments between the following pairs of points:

a. (0,0) and (5,3).
b. (1,4) and (4,8).
c. (-2,2) and (3,-4).
d. (-1,0) and (-3,-2).
e. (-2,-3) and (-2,3).

2. If a square is placed with two of its sides along the x- and y-axes, what are the slopes of each of its diagonals.

3. If scalene Δ ABC is placed with $\overline{AB}$ along the x-axis which of the following lines has no slope? $\overline{AB}$, the median to $\overline{AB}$, the altitude to $\overline{AB}$, the angle bisector of $\angle C$.

C. 1. Determine the distance between each pair of points:

a. (1,4) and (2,3).
b. (-1,0) and (-9,15).
c. (a,b) and (-a,-b).

2. If three of the vertices of a rectangle are at (0,1), (5,1) and (5,4) what is the length of a diagonal of the rectangle.

3. The vertices of a trapezoid are (0,0), (a,0), (b,c) and (d,c). What is the length of the segment joining mid-points of its non-parallel sides?

D. 1. A triangle has vertices A(0,0), B(12,0) and C(9,6). What is the equation of the median to side $\overline{AB}$?

2. Of the following equations which pairs of lines are a. parallel, b. coincident, c. intersecting, d. perpendicular.

(1) $3y = 6x - 3$.

(2) $y - 2x = 5$.

(3) $y = 2 - 2x$.

(4) $2y + 1 = x$.

3. A right triangle has vertices (0,0), (m,0), (0,n). What is the equation of the median which passes through the origin?

E. 1. Using coordinate geometry prove that the mid-point of the hypotenuse of a right triangle is equidistant from the vertices.

2. Show that the points A, B, C, D whose coordinates are (2,3), (4,1), (8,2), (6,4) are vertices of a parallelogram. Show that the figure formed by joining the mid-points of the sides of ABCD is a parallelogram.

3. Prove by coordinate geometry the theorem: If a line parallel to one side of a triangle bisects a second side, then it also bisects the third side.

Answers

A. 1. The origin.

2. I, IV, II.

3. (4,0) or (-4,0).

4. $(\sqrt{3},1)$.

B. 1. a. $\frac{3}{5}$. b. $\frac{4}{3}$. c. $\frac{-6}{5}$. d. 1.

e. The line is vertical and has no slope.

2. 1, -1.

3. The altitude to $\overline{AB}$.

C. 1. a. $\sqrt{2}$. b. 17. c. $2\sqrt{a^2 + b^2}$.

2. $\sqrt{34}$.

3. $\frac{1}{2}(|a| + |b - d|)$.

D. 1. $y = 2(x - 6)$.

2. a. (1) and (2).

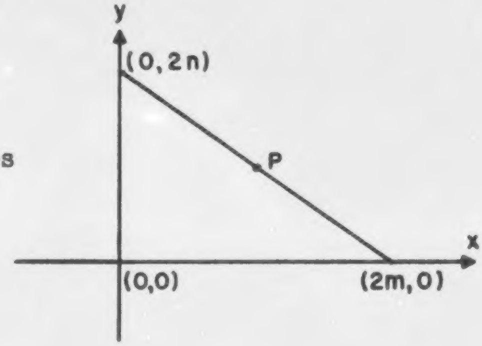
b. None.

c. (1) and (3); (1) and (4); (2) and (3);
(2) and (4); (3) and (4).

d. (3) and (4).

3. $my = nx$.

E. 1. Take a coordinate system as shown, with vertices (0,0), (2m,0), (0,2n). Then mid-point P of hypotenuse has coordinates (m,n). Distance of P from each vertex is $\sqrt{m^2 + n^2}$.



2. Slope $\overline{AB} = -1 =$ slope $\overline{CD}$.

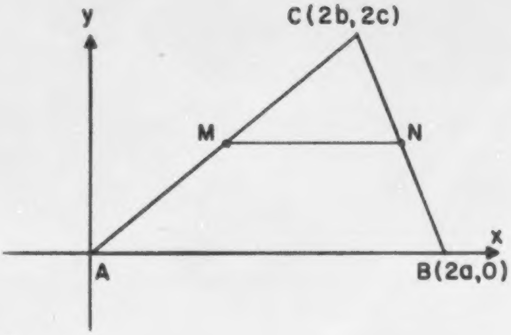
   Slope $\overline{AD} = \frac{1}{4} =$ slope $\overline{BC}$.

   Hence, $\overleftrightarrow{AB} \neq \overleftrightarrow{CD}$, so that $\overline{AB} \parallel \overline{CD}$.

   Likewise $\overline{AD} \parallel \overline{BC}$.

   The mid-points of the sides taken in order are $(3,2)$, $(6,1\frac{1}{2})$, $(7,3)$ and $(4,3\frac{1}{2})$. Slopes of sides of the figure formed by joining these mid-points are $-\frac{1}{6}$ for each of one pair of sides and $\frac{3}{2}$ for each of the other pairs. Hence, this figure also is a parallelogram.

3. Select a coordinate system in such a way that the vertices are $A(0,0)$, $B(2a,0)$, $C(2b,2c)$. Let M be mid-point of $\overline{AC}$, $\overline{MN} \parallel \overline{AB}$. Then $M = (b,c)$. Slope $\overline{MN} = 0$. Hence, equation $\overleftrightarrow{MN}$ is $y = c$.



   Equation $\overleftrightarrow{BC}$ is $y = \frac{c}{b - a}(x - 2a)$.

   Solving these equations we find $N = (a + b,c)$. Hence, (from mid-point formula) N is the mid-point of $\overline{BC}$.

FACTS AND THEORIES

Science today is playing an increasingly important part in the life of the individual. No one can claim to be truly educated unless he has a reasonable understanding of the facts and methods of science. This does not mean that we must all become nuclear physicists, nor that we must spend all our time reading books and attending lectures on the latest collection of particles discovered by the physicists. But it does impose on us the obligation to learn enough of the facts of modern science to provide a foundation for understanding. It does imply an intelligent selection of material to be learned.

We, as educators, are especially obligated to make such a selection for our students. They come to us with a miscellaneous hodgepodge of disjointed facts and pseudo-facts, gleaned from newspapers, magazines, books, and other sources. We must help them -- with our own limited information -- to straighten out their ideas, to build a reasonable conceptual structure upon which they can hang new facts, to distinguish between that which is significant and that which is not, and, perhaps most important of all, to understand how new knowledge is acquired. If pursued to the extreme, this last goal would lead us to the far reaches of epistomology and scientific method, which have been the subjects of many weighty tomes written by scholars over many lifetimes, and about which the last word has certainly not been uttered. But to dismiss this topic entirely as being too subtle for the immature minds of our students is to deny them the opportunity of becoming a little more mature in our classrooms.

What should be the aims of the mathematics teacher, in the light of what we have just said?

Certainly we should help the student to become acquainted with the facts of mathematics by working with them. We agree that our subject is an essential tool in science and in daily life, and that the student should acquire a working facility in it. Therefore we teach him arithmetic, elementary algebra, intuitive geometry in the lower grades, advanced algebra, synthetic and analytic geometry, possibly calculus and other topics in the higher grades.

It would be difficult, however, to defend the teaching of all these subjects on the grounds of utility alone. No one pretends, for example, that it is of practical importance that the bisector of an angle of a triangle divides the opposite side in the way that it does. We proceed, then, to the second aim, of developing in the student an appreciation of clear, logical reasoning as exemplified in mathematics, and an ability to transfer this type of reasoning to other situations. We have been moderately, though not eminently, successful in this respect in the past. Whether our present efforts will tend to further this objective remains to be seen. We certainly hope so.

A third aim, which has been receiving more attention of late, is to develop in the student an understanding of the structure of mathematical systems. We are beginning to speak of closure, commutativity, distributivity and so on in dealing with number systems, and -- still too timidly, perhaps -- of the axiomatic nature of geometry.

This third aim is closely related to the broader one mentioned earlier, of helping the student to understand how new knowledge is acquired, how man learns about the physical world, how he constructs, develops and tests theories about the physical, biological, social, and economic aspects of life around him. Let us address ourselves briefly to these questions.

Whether we recognize it or not, theory plays an indispensable role in our study of any field whatsoever. The acts of naming, classifying, and generalizing are conceptual in nature. Even emotional reactions to stimuli depend on a structuring of experience. The real world -- whatever that may mean -- reaches us only by constructing a conceptual world to correspond to it. In setting up a particular discipline, it is not necessary, however, to refer back always to the primary data supplied by our senses. The raw material for a theory at one stage may be the conceptual world of a previous stage. For example, the classical geometry of various surfaces in three dimensions may be taken as the jumping-off place for a study of abstract metric spaces, and we would then abstract from this classical geometry, testing our new theory against it. In every case, then, we operate simultaneously in two different "planes." One is the primary, intuitive plane, containing the raw data from which our theory will be abstracted. This, following Bridgman, we call the "P-plane." The second is the conceptual plane, the "C-plane." Initially, the C-plane is empty, waiting to be filled with the concepts and relations which we construct.

We have complete freedom with respect to the concepts and relations which we choose to insert in the C-plane, so long as we do not assert any connection between it and the P-plane. Naturally, we hope eventually to set up a correspondence between the two planes, and this hope guides our constructions and our choice of language. Logically, there is no necessity to make the language in the C-plane correspond to that of the P-plane, and in order to avoid confusion it might be better to use different terms entirely. For example, the "points," "lines," and "planes" of axiomatic geometry (the C-plane) might be replaced by other terms which have not been preempted in physical geometry (the P-plane). But once the formal distinction between the two planes and their languages has been established and understood, there is a psychological advantage to be gained from the use of the same terms, for the proposed correspondence is then transparently indicated. Thus, we know that the geometrical "point" is meant to correspond to the physical point, the geometrical "line" to the physical line, and so on. We can intuit, conjecture, and then perhaps prove theorems in the C-plane by peeking over into the P-plane at the corresponding "facts," arrived at by experiment there. For example, the concurrence of the medians of a triangle could be guessed from drawing a number of physical triangles and their medians on a piece of paper. This type of experience is extremely valuable and constitutes an important psychological adjunct to mathematical discovery. It must be pointed out carefully, though, that formal proof in the C-plane is necessary. Furthermore, the logical conclusion to be drawn from this combined guessing and proving process is not that we have made the geometrical theorem more certain by experimental verification. The truth of the theorem has been established (in the C-plane) with complete certainty by logical deduction from the axioms. Rather, our feeling of satisfaction on seeing that the theorem works out on paper should stem from the confirmation of the correspondence between the two planes. What we do tend to establish by such empirical tests is the adequacy of our postulate system to bring about a close correspondence.

Consider for example, what our situation would be if we had in our system all of the postulates of Euclidean geometry except for the parallel postulate. Suppose, then, that we measured the angles of many triangles and found, within the limits of experimental error, that the sum of the measures of the angles was 180. Then, passing to the C-plane, we attempted to prove the corresponding result as a theorem, and of course failed. The correct conclusion to draw would be that (a) we were not clever enough to find a proof, or (b) that our axiom system was not adequate for the purpose. Historically, it was the belief that (a) was the only possibility, together with an imperfect understanding of axiomatics, that delayed the development of non-Euclidean geoemtry. Eventually, of course, this very problem led to our present deeper understanding of the connection between fact and theory.

What are the considerations that govern our choice of undefined elements and relations and unproved propositions (axioms, postulates)? Certainly we want our system to be consistent: a proposition and its contradiction should not both be provable in the system. If we regard our axioms as inputs and our theorems as outputs, then economy and fruitfulness are desirable as increasing output per unit input. Of course, this analogy is not to be taken too seriously, but it indicates why we should not postulate everything. Unfortunately, some geometry texts nowadays go to the extreme of setting down fifty or more postulates. There is nothing logically wrong with this, but it militates against economy, elegance, intuitiveness, simplicity, and ease of verification in a particular interpretation -- properties that are certainly desirable.

One property that we have not mentioned is that of being categorical. This means that every two concrete interpretations (models) of the system will be essentially the same: it is possible to set up a one-to-one correspondence between the elements and relations of the two interpretations, so that they may be regarded as identical except for the names assigned to the elements and relations. The two models are then said to be isomorphic. If we start with a particular P-plane and wish to describe it completely by means of an axiom system, without permitting any non-isomorphic models, then we try to make our system categorical. This is the case with Euclidean geometry or the real number system.

Sometimes we reap an unexpected harvest from the construction of a categorical system. We may find two apparently different interpretations, and can then conclude that they are essentially identical because the system is categorical. Any theorem which holds in one model is then sure to hold in the other. An example is the pair of models M_1, consisting of the real numbers under addition, and M_2, consisting of the positive real numbers under multiplication. The one-to-one correspondence $M_1 \longleftrightarrow M_2$ is established by the exponential function (from M_1 to M_2) and the logarithm (from M_2 to M_1). Another example is the pair of physical processes, diffusion of a gas and heat-flow, both being governed by the same differential equation. Still another example is the isomorphism of Euclidean plane geometry with the collection of all real-number pairs. This isomorphism allows us to solve geometrical problems by means of algebra, and vice versa.

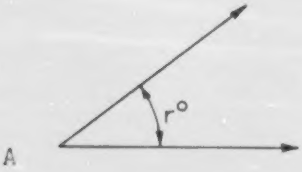
At other times, we find it more profitable to make our system non-categorical. This is true when we have several P-planes which bear some resemblance to each other. If we can construct a suitable C-plane so that each of the P-planes is an interpretation of it, then anything we prove in the C-plane will hold in all of its non-isomorphic models. This happens, for example, in the case of group theory. It also happens when we state a few, but not all of the axioms of Euclidean geometry. In this case our theorems, being provable, say, without the parallel postulate, must hold also for all geometries satisfying the stated axioms. There is no reason to hide this desirable state of affairs from our students, for fear of violating their intuitions about space. Rather, we should regard such occasions as valuable opportunities for teaching an important lesson.

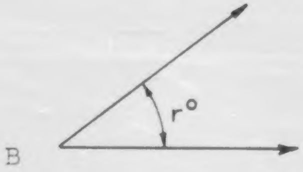
Our discussion here has been far from exhaustive. We hope that it has served the purpose of pointing to a desirable and sometimes neglected goal in education, and that it has indicated how we, as teachers of mathematics, can approach this goal.

EQUALITY, CONGRUENCE, AND EQUIVALENCE

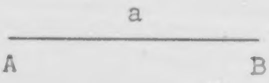
1. Angles and Segments.

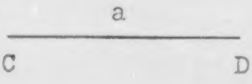
In describing the relation of "equality" between angles and segments, this book departs from common usage. Before explaining why this has been done, let us first note quickly how the new usage compares with the old. Suppose we have two angles with the same degree measure r, like this:





and two segments of the same length, like this:





In these two instances, the facts are plain. They would be reported in the following ways, in the old and new terminologies.

In Words		In Symbols	
Old	New	Old	New
The angles are equal.	The angles are congruent.	$\angle A = \angle B$	$\angle A \cong \angle B$ (or $m\angle A = m\angle B$)
The segments are equal.	The segments are congruent.	$\overline{AB} = \overline{CD}$	$\overline{AB} \cong \overline{CD}$ (or $AB = CD$).

From the table it is plain that the new usage is not complicated. We have simply substituted one word for another, and one symbol for another. Of course, even simple changes should be made only for good reasons; they go against everybody's habits, and cause much more trouble at first than their simplicity would suggest. We believe that there are very good reasons for the use that we have made of the word congruence. Following is an explanation of what these reasons are.

2. Various Kinds of Equality.

The word "equals" is commonly used in mathematics in at least this many different senses:

(1) When we write $2 + 4 = 3 + 3$, we mean that the number $2 + 4$ and the number $3 + 3$ are exactly the same number (namely, 6). Here "equals" means "is the same as."

(2) When we say that two angles are equal, we mean that they have the same measure, or the same shape.

(3) Two circles are equal if they have the same radius.

(4) Two segments are equal if they have the same length.

(5) Two triangles are equal if they have the same area.

(6) Two polyhedrons are equal if they have the same volume.

These uses of "equals" divide sharply into three groups.

(I) The first meaning ("is the same as") stands entirely alone. This is the logical identity. It arises in all branches of mathematics, including geometry.

(II) "Equality" expresses the same basic idea for angles, circles, and segments, in (2), (3), and (4). It means in each case that the first figure can be moved so as to coincide with the second without stretching. (For a fuller explanation, see Appendix VIII, on Rigid Motion.) This idea is geometric, and is one of the most basic ideas in geometry. Applied to triangles, it is always described as congruence and not as equality.

(III) "Equality" to mean equal areas or equal volumes, as in (5) and (6), implies that two things are equal if they contain the same amount of "stuff."

These are the three main ideas involved. We notice that the words and the ideas overlap both ways. Not only is the word "equals" used in two widely different senses, but the basic idea involved in (2), (3), and (4) is expressed by two apparently unrelated words.

Obviously students can and do learn to keep track of what is meant, even when the words and the ideas overlap in this way. All of us learned to do this, when we were in the tenth grade. The whole thing becomes easier to learn, however, and easier to keep track of, if the words match up with the ideas in a simpler and more natural way. This can be done as follows:

(I) We can agree to write "=", and say "equals," only when we mean "is the same as." (This is the standard usage in nearly all of modern mathematics.)

(II) We already have a word to express the idea that one triangle can be made to coincide with another; we say that they are congruent. We can use the same word to express the same idea when we are talking about angles, circles or segments.

(III) When we want to convey the idea that two triangles have the same area, we can simply say that they have the same area.

Notice that if we do this we have not introduced any new words into the language of geometry. We are not trying to be technical. All that we are trying to get at is a situation in which the familiar and available words correspond in a natural way to the familiar and basic ideas. The correspondence looks like this:

(I) =, between any two things whatever, means "is the same as."

(II) $\cong$, between any two geometric figures whatever, means that one can be moved so as to coincide with the other.

(III) Equality of area, equality of volume, and so on, are to be described explicitly as such.

All this is straightforward language. We believe that your students will find it easy to learn and easy to use.

3. Equivalence Relations.

All the uses of "equals," in mathematics or otherwise, involve the notion of two things being alike in some respect. The particular respect to be considered may be made explicit, as in usage (5) above, or it may not, as in "All men are created equal." As mentioned above, mathematicians have pretty generally agreed to use the word to mean "alike in all respects"; that is, identical. Instead of the other usage they speak of an "equivalence relation." A relation between pairs of objects, from some given set, is called an equivalence relation if it has the following three properties:

(1) It is reflexive. That is, any object of the set is equivalent to itself.

(2) It is symmetric. That is, if A is equivalent to B, then B is equivalent to A.

(3) It is transitive. That is, if A is equivalent to B, and B is equivalent to C, then A is equivalent to C.

In a mathematical development we may use several different kinds of equivalence relations. To keep them separate we give them different names and different symbols. In our geometry we have used the following equivalence relations.

(a) Identity. The relation "is the same as" is easily seen to satisfy the three properties listed above. The word "equal" and the symbol "=" are reserved for this equivalence relation.

(b) Congruence. Here again, the properties are easily checked. (Refer to the talk on Congruence for a general treatment.) The symbol is " $\cong$ ".

(c) Similarity. Here again we have an equivalence relation, denoted by "$\sim$".

(d) We have not introduced any special notation for "equality of area," or "equality of volume," but each of these relations is reflexive, symmetric and transitive. We could, if it were convenient, introduce words and symbols for these equivalence relations.

Such insistence on exactitude of language and symbolism may sometimes seem mere quibbling, but it is on such extreme carefulness that modern mathematics is based.

4. Classification and Functions.

Equivalence relations are connected closely with another concept which is important in mathematics. This is classification. The connection is as follows.

Suppose we have an equivalence relation $\approx$ defined for a certain set S. We can then classify the elements of S into disjoint classes (i.e. no two classes intersect) $S_1, S_2, \ldots,$ by putting into the same class all elements which are equivalent to each other. Conversely, suppose that we have a classification of S into disjoint classes. Then we can define an equivalence relation by saying that a is equivalent to b if and only if a and b are in the same class. These two constructions

$$\text{Equivalence} \longrightarrow \text{Classification},$$
$$\text{Classification} \longrightarrow \text{Equivalence},$$

are inverses of each other. If we start with an equivalence, pass to its classification, and then pass from this classification to its induced equivalence, we end up with the same equivalence. Similarly, if we start with a classification, form the induced equivalence, then form its induced classification, we end up with the same classification.

An example may make this clearer. Suppose S is the set of all polygons. Let us define $\approx$ among polygons by saying that $P_1 \approx P_2$ if P_1 and P_2 have the same number of sides. (This relation $\approx$ obviously is reflexive, symmetric and transitive.) The induced classification is then into triangles, quadrilaterals, pentagons, hexagons, ..., n-gons, ... If we start with this classification, its induced equivalence is: $P_1 \approx P_2$ if P_1 and P_2 are in the same class, i.e., if they are both n-gons (for the same n). This is the same as the original equivalence.

Notice that in this example, our classification was by means of a unique number attached to each polygon, namely the number of sides. Whenever we have a unique number attached to each object of a set S, we have a numerical function f(a). Thus, every numerical function induces a classification: each class consists precisely of those elements a with the same functional value f(a). As another example let S be the set of angles and let $f(a) = m\angle a$. The corresponding equivalence relation is then our familiar congruence $\cong$, between angles.

On the other hand, not every equivalence relation is easily characterized by a function. If S is the set of triangles it is hard to see how the similarity relation, $\sim$, or the congruence relation, $\cong$, can be associated with a function. As a matter of fact this can be done, but the methods involved are well beyond elementary mathematics, as well as being highly artificial.

THE CONCEPT OF CONGRUENCE

Congruence is a rich and complex idea with many ramifications in geometry - there really is nothing quite like it in algebra. It applies to figures of all kinds - segments, angles, triangles, circular arcs, polygons, truncated pyramids - in fact to any conceivable figure. It plays an essential role in the theory of geometric measure of length, area and volume - it is intimately related to the important concept of rigid motion.

We will examine carefully the conventional theory of congruence and the related theory of linear measure. This will be contrasted with the theory of congruence adopted in our text. Finally we treat the concept of congruence for general figures and its relation to the idea of rigid motion.

I. The Conventional Theory of Congruence and Linear Measure

I-1. Congruence in terms of size and shape. The term congruence immediately calls to mind the famous dictum: Two figures are congruent if they have the same size and the same shape. Certainly this statement emphasizes the basic intuitive or informal idea that if two figures are congruent, one is a "replica" of the other. Also it points up the important property that if we know two figures to be congruent we can infer that they have the same area (or volume) and that they are similar.

But this is not the essential issue. It is: Does our dictum *define* congruence? Is it really a formal definition of the term congruence in terms of more basic ideas? Clearly the answer is no. For the notions size and shape are *more* complex than congruence. In order to measure (or define) size (area or volume) we try to find out how many *congruent* replicas of a basic figure (for example, square or cube) "fill out" a given figure. So actually it would be more natural and simple to base the theory of size (and shape) on the idea of congruence rather than the reverse.

I-2. Congruence in terms of rigid motion. But there are other "definitions" of congruence which we must discuss - consider the famous, "Two figures are congruent if they can be made to coincide by a rigid motion". Let us analyze this. Conceived concretely, say in terms of two paper heart-shaped valentines, it affords an excellent illustration of the intuitive idea of congruence and emphasizes again that one is a "replica" of the other. But this illustration, like most physical situations, does not have the precision required for an abstract mathematical concept. Surely we would have to pick up the first valentine and move it with almost infinite gentleness to prevent bending it slightly when getting it to coincide with the second one. And how could we be certain of perfect coincidence of the two valentines? Wouldn't this require perfect eyesight? It is clear that this "definition" interpreted concretely gives us a physical approximation to the abstract idea of congruence but doesn't define it. Moreover it is not even applicable in many physical situations: you hardly could get two "congruent" billiard balls to coincide by a rigid motion.

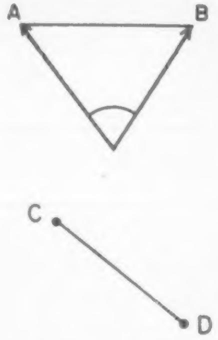
Should we then conclude that the idea of rigid motion is essentially physical and can not be mathematicized as an abstract geometrical concept? Definitely not. Mathematicians are ingenious and clever people and it might be a mistake to decide beforehand that they could not construct a precise abstraction from a given physical idea. Most familiar mathematical abstractions had their origin in concrete physical situations - certainly geometry had its origin in practical problems of surveying the heavens and the earth.

Let us table for the present the question of whether we can form an abstract geometrical theory of rigid motions. It would seem that a treatment of congruence based on a logically satisfactory theory of rigid motion could not be elementary and would hardly be suitable for a first course. In any case, without deeper analysis, the second "definition" is not a definition at all and might more properly be considered a statement of a property which rigid motions should have: namely, that any rigid motion transforms a figure into a congruent one.

I-3. *Another definition*. Consider and criticize a third suggested "definition": Two (plane) figures are congruent if a copy of the first made on tracing paper can be made to coincide with the second.

I-4. *Congruence of segments*. Since our three "definitions" do not define congruence we must probe more deeply. Here, as so often in solving problems, the imperialist maxim, "Divide and conquer", is very helpful. Instead of tackling the concept of congruence in its most complex form, that is, for arbitrary figures, let us begin by considering a simple special case. A line segment -- or as we shall call it, a segment -- is one of the simplest and most important geometric figures. We naturally begin by considering congruence of segments.

Let us recall how this is treated in Euclid or in the conventional high school geometry course. Congruent segments, usually called equal segments, are conceived as "replicas" of each other, in general with different locations in space. Congruent segments may coincide or be identical but they don't have to. If segments $\overline{AB}$ and $\overline{CD}$ are congruent we may interpret this concretely to mean $\overline{AB}$ and $\overline{CD}$ are "caliper equivalent" - that is, if a pair of calipers is set so that the ends coincide with A and B, then, without changing the setting, the ends of the calipers can be made to coincide with C and D.



I-5. *Basic properties of congruence of segments*. What is the *logical significance* of congruence of segments in Euclid? Actually it is taken to be an undefined term. More precisely, using the notation $\overline{AB} \cong \overline{CD}$, congruence is a basic relation $\cong$ between the segments $\overline{AB}$ and $\overline{CD}$ which we do not attempt to define. We study it (as always in mathematics) in terms of its basic properties which are formally stated as postulates. Some of these postulates, which are not explicit in Euclid or in most geometry texts are:

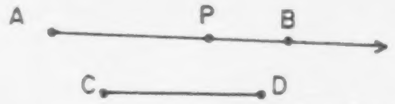
(1) (Reflexive Law) $\overline{AB} \cong \overline{AB}$;

(2) (Symmetry Law) If $\overline{AB} \cong \overline{CD}$ then $\overline{CD} \cong \overline{AB}$;

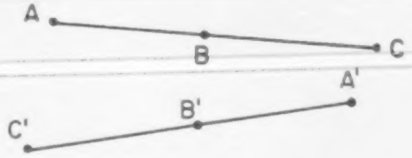
(3) (Transitive Law) If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$ then $\overline{AB} \cong \overline{EF}$.

That is, congruence of segments satisfies the three basic properties of equality or identity and so is an example of an equivalence relation. We must not assume that congruence means identity, since distinct segments can be congruent.

(4) (Location Postulate) Let $\overrightarrow{AB}$ be a ray and let $\overline{CD}$ be a segment. Then there exists a unique point P in $\overrightarrow{AB}$ such that $\overline{AP} \cong \overline{CD}$.



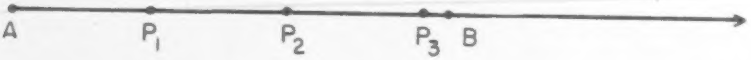
(5) (Additivity Postulate) Suppose $\overline{AB} \cong \overline{A'B'}$, $\overline{BC} \cong \overline{B'C'}$, B is between A and C and B' is between A' and C'. Then $\overline{AC} \cong \overline{A'C'}$.



We insert a few words on the important mathematical idea of equivalence relation. The most basic example of an equivalence relation and the one which suggests the concept is the relation equality or identity. Equivalence relations abound in geometry, for example, congruence of figures or similarity or equivalence of figures. (For a discussion of equivalence relations see the Talk on Equality, Congruence, and Equivalence.)

I-6. Theory of linear measure. Segments are geometric figures, not numbers. But they can be measured by numbers -- they do have lengths. In the conventional high school treatment it is assumed with little discussion that lengths of segments can be defined as real numbers. We indicate how to do this. Although the result is familiar, the process is complex and subtle and requires for its complete justification additional postulates. However, Postulates (1), ..., (5) above are sufficient for an understanding of the process.

We begin by choosing a segment $\overline{UV}$ which will be unchanged throughout the discussion (a so-called "unit" segment). Now given any segment $\overline{AB}$ we want to measure $\overline{AB}$ in terms of $\overline{UV}$. This involves a "laying-off" process. We take the ray $\overrightarrow{AB}$ and lay-off



$\overline{UV}$ on it repeatedly, starting at A. Speaking precisely, there is a point P_1 in $\overrightarrow{AB}$ such that $\overline{UV} \cong \overline{AP_1}$. Similarly, we can show that there is a point P_2 in $\overrightarrow{AB}$ such that (a) $\overline{UV} \cong \overline{P_1P_2}$ and (b) P_1 is between A and P_2. For convenience we write condition (b) as (AP_1P_2). Continuing, there is a point P_3 such that $\overline{UV} \cong \overline{P_2P_3}$ and $(P_1P_2P_3)$. By this process we develop a sequence of points $P_1, P_2, \ldots, P_n, \ldots$ on $\overrightarrow{AB}$ such that

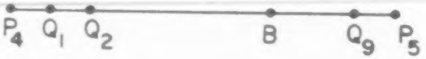
(1) $\overline{UV} \cong \overline{P_1P_2} \cong \overline{P_2P_3} \cong \ldots \cong \overline{P_{n-1}P_n}$,

(2) $(AP_1P_2), (P_1P_2P_3), \ldots, (P_{n-2}P_{n-1}P_n)$.

Intuitively (1) and (2) say that $\overline{UV}$ is laid-off on AB n times in a given direction - but note how very precisely and objectively (1), (2) say this, avoiding the somewhat vague terms "laying-off" and "direction". From another viewpoint we are laying the basis for a coordinate system on the line by locating precisely the points $P_1, P_2, \ldots, P_n, \ldots$ which are to correspond to the integers $1, 2, \ldots, n, \ldots$.

Now what has this to do with the measure of $\overline{AB}$? Clearly we must learn how B is related to the points $P_1, P_2, P_3, \ldots$. In the simplest case one of these might coincide with B, for example, $P_3 = B$. Then of course we define the measure of $\overline{AB}$ to be 3.

I-7. Refinement of the approximation process. You may ask, "Did we have to go through this elaborate process to explain that if the "unit" segment $\overline{UV}$ exactly covers $\overline{AB}$ three times, then the measure of $\overline{AB}$ is 3?" Disregarding the importance of making the idea "exactly covers" mathematically precise, observe that the process helps us to define a measure for $\overline{AB}$ in the more general and difficult case when no one of the points P_1, P_2, ... coincides with B. For suppose B falls between two consecutive points of our sequence, say (P_4BP_5). Clearly then we will have to assign to $\overline{AB}$ a measure x such that $4 < x < 5$. In other words we have set up a general process which enables us at least to determine an approximation to the measure of $\overline{AB}$, that is to find lower and upper bounds for it.



We do not complete the discussion but indicate how it proceeds. To fix our ideas, suppose (P_4BP_5). To get a better idea of what the measure of $\overline{AB}$ should be we subdivide P_4P_5 into ten congruent subsegments and proceed as above. Precisely, we set up a subsidiary sequence of points Q_1, ... Q_9 which divide $\overline{P_4P_5}$ into ten congruent subsegments. That is, we require

$$\overline{P_4Q_1} \cong \overline{Q_1Q_2} \cong \overline{Q_2Q_3} \cong \ldots \cong \overline{Q_9P_5}$$

and

$$(P_4Q_1Q_2),\ (Q_1Q_2Q_3),\ \ldots,\ (Q_8Q_9P_5).$$

If B were to coincide with one of Q_1, Q_2, ..., Q_9, say $B = Q_6$, we assign to $\overline{AB}$ the measure 4.6. If B falls between two of the Q's, say (Q_6BQ_7), we require that x, the measure of $\overline{AB}$, satisfy

$$4.6 < x < 4.7.$$

In the latter case we repeat the process by subdividing $\overline{Q_6Q_7}$ into ten congruent subsegments and proceed as before.

I-8. *The definition of linear measure*. Clearly we have a complex process (though a refinement of a simple idea) which will assign to segment $\overline{AB}$ a definite decimal, terminating or endless. This decimal we define to be the *measure* or *length* of $\overline{AB}$.

I-9. *Basic properties of linear measure*. We write the measure of $\overline{AB}$ ($\overline{UV}$ still being fixed) as $m(\overline{AB})$. Observe that we really have here a function $\overline{AB} \longrightarrow m(\overline{AB})$ which associates to each segment a unique positive real number. What are the basic properties of this "measure" function? They are easily grasped intuitively:

(1) $m(\overline{AB}) = m(\overline{A'B'})$ if and only if $\overline{AB} \cong \overline{A'B'}$ - that is, congruent segments and only congruent segments have equal measures;

(2) If (ABC) then $m(\overline{AB}) + m(\overline{BC}) = m(\overline{AC})$ - that is, measure is *additive* in a natural sense;

(3) $m(\overline{UV}) = 1$ - that is, the measure of the unit segment is unity.

Notice that (2) is a clear and useful form of the vague statement, "the whole is the sum of its parts".

We summarize in a theorem which can be deduced from a suitable set of postulates for Euclidean Geometry:

Theorem. Let the segment $\overline{UV}$ be given. Then there exists a function which assigns to each segment $\overline{AB}$ a unique positive real number $m(\overline{AB})$ satisfying (1), (2), (3) above.

I-10. *Uniqueness of measure function*. We naturally ask if there is just one measure function? Clearly not. For the function must depend on the choice of the unit segment $\overline{UV}$. To be specific, suppose we take as a new unit segment, $\overline{UM}$, where M is the mid-point of $\overline{UV}$ (that is $\overline{UM} \cong \overline{MV}$ and (UMV)). Then according to our theorem there will be a measure function; let us call it m' (since we have no right to assume it is the same as the original measure function) such that $m'(\overline{UM}) = 1$. We see quickly that $m'(\overline{UV}) = 2$;

further it can be shown $m'(\overline{AB}) = 2m(\overline{AB})$ for any segment $\overline{AB}$. This is a formal statement of the trivial seeming fact that "halving the unit of measurement doubles the measure". A corresponding result holds in general:

Theorem. If m, m' are two measure functions on the set of all segments, then

$$m'(\overline{AB}) = k\cdot m(\overline{AB})$$

where k is a fixed positive real number.
In the preceding example we had $k = 2$. Of course k need not be an integer - it can be any positive real number, rational or irrational. As a related example consider the corresponding situation in the measure of angles: The radian measure of an angle is $\frac{\pi}{180}$ times the degree measure of the angle.

Summary: Any two measure functions on the set of all segments are proportional.

What does this mean for the development of the theory of measurement of segments? It says in effect that it doesn't matter which measure function we choose, since making a different choice would only multiply all measures by a constant. Thus, in conventional geometrical theory, we fix a unit $\overline{UV}$ at the beginning, determine a corresponding measure function, and thereafter use this measure function as if it were the only possible one. And instead of saying precisely the measure of $\overline{AB}$ in terms of unit $\overline{UV}$, we say simply the measure of $\overline{AB}$, and forget about $\overline{UV}$. The situation in everyday life is quite different - we employ measure functions based on a variety of units: inches, light years, millimeters, miles.

We close this part of our discussion by observing that the distance between A and B is merely defined to be the measure of $\overline{AB}$. Sometimes we want to refer to the distance between A and A itself. This we take to be zero. A separate definition is required for this case, since we may not refer to the segment $\overline{AB}$ unless we know $A \neq B$.

Query. Was it necessary to use the integer ten in the subdivision process? Would others work? Could the process be simplified by making a different choice?

II. Congruence Based on Distance

In this part we discuss the treatment of congruence adopted in the text, contrasting it with the conventional one. The point of departure is to "reverse" the conventional treatment and define congruence in terms of distance. This enables us to use our knowledge of the real number system early in the discussion - it leads to a new treatment of the important geometric relation, betweenness, and a new way of conceiving segments and rays.

II-1. The student's viewpoint. The conventional treatment, in brief, begins with an undefined notion of congruence of segments and deduces the existence of a distance function from a suitable set of postulates. The high school student - in studying this treatment - somehow absorbs the idea that segments (and angles) can be measured by numbers, and is permitted to apply his knowledge of algebra whenever it is convenient.

II-2. The Distance Postulate. Since the student thinks of segments and angles as measurable by numbers and it is hopeless to prove this at his level from non-numerical postulates, it seems most reasonable to make the existence of a measure function or distance a basic postulate which is used consistently throughout the course. So we adopt

Postulate 2. (The Distance Postulate.) To every pair of different points there corresponds a unique positive number.

If the points are P and Q, then the distance between P and Q is defined to be the positive number of Postulate 2, denoted by PQ.

Don't read into this more than it says - it is a very weak statement. Notice that it doesn't state a single property of distance - merely that there is such a thing. In particular it doesn't say anything about lengths of segments - in fact we don't even have segments at this stage of our theory.

II-3. *The Distance Postulate causes a change in viewpoint.* This may seem strange, but it isn't. Most texts begin with a discussion of points and lines in a plane, including such basic ideas as *segment* and *ray*. As in Euclid these ideas essentially are taken as undefined. But having adopted the Distance Postulate we *can* define them. This is an important - and unforeseen - consequence of the Distance Postulate: We don't get just Euclid with the theorems rearranged, but new insights into the basic geometric ideas and a new way of inter-relating them.

II-4. *"Between" and "Segment" as defined terms.* How then can we *define* segment in terms of the basic terms point, line, plane? It is easy to do this using the additional notion of a point being *between* two points. Having adopted Postulate 2, the idea of distance is at our disposal and we can define betweenness so:

Definition. Let A, B, C be three collinear points. If $AB + BC = AC$ we say B is *between* A and C, and we write (ABC).

We now define segment in terms of betweenness.

Definition. Let A, B be two points. Then *segment* $\overline{AB}$ is the set consisting of A and B together with all points that are between A and B. A and B are called *endpoints* of $\overline{AB}$. Further we define $m(\overline{AB})$, the *measure* or *length* of $\overline{AB}$, merely to be the number AB.

That is, the length of a segment is merely the number which is the distance between its endpoints. The contrast with conventional theory is striking: There congruence of segments is basic and a difficult argument is needed to prove the existence of a measure function - here distance is basic and the proof of the existence of a measure function is trivial.

II-5. *Congruence of segments by Definition.* Now it is absurdly easy to *define* congruence of segments.

Definition. $\overline{AB} \cong \overline{CD}$ means that the lengths of $\overline{AB}$ and $\overline{CD}$ are equal, that is $AB = CD$.

Formally what we have done is just this. We took the basic property relating congruence and measure ((1) of Section I-9)

$$m(\overline{AB}) = m(\overline{CD}) \text{ if and only if } \overline{AB} \cong \overline{CD},$$

which is a theorem in the conventional treatment, and adopted it as a definition in our treatment. There, segments which were congruent were proved to have the same measure - here, segments which happen to have the same measure are called congruent.

II-6. Properties of congruent segments. Does congruence of segments, as we have defined it, have the properties we expect? We see quickly that $\cong$ is an equivalence relation, that is

(1) $\overline{AB} \cong \overline{AB}$;

(2) If $\overline{AB} \cong \overline{CD}$ then $\overline{CD} \cong \overline{AB}$;

(3) If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$ then $\overline{AB} \cong \overline{EF}$.

These merely say

(1') $AB = AB$;

(2') If $AB = CD$ then $CD = AB$;

(3') If $AB = CD$ and $CD = EF$ then $AB = EF$,

which are the basic properties of equality of numbers.

Further we have

(5) Suppose $\overline{AB} \cong \overline{A'B'}$, $\overline{BC} \cong \overline{B'C'}$, (ABC) and $(A'B'C')$. Then $\overline{AC} \cong \overline{A'C'}$.

To prove this we have

$$AB = A'B',$$

$$BC = B'C',$$

so that

$$AB + BC = A'B' + B'C'.$$

The betweenness relations yield

$$AB + BC = AC, \qquad A'B' + B'C' = A'C',$$

and we get

$$AC = A'C' \quad \text{or} \quad \overline{AC} \cong \overline{A'C'}.$$

Thus several of Euclid's (or Hilbert's) Postulates for congruence reduce, in our treatment, to elementary properties of real numbers.

II-7. The Ruler Postulate. You may wonder if we can also derive from Postulates 1 and 2, the Location Property: ((4), Section I-5):

Let $\overrightarrow{AB}$ be a ray and let $\overline{CD}$ be a segment. Then there exists a unique point P in $\overrightarrow{AB}$ such that $\overline{AP} \cong \overline{CD}$.

The answer is - with a vengeance - no. On the basis of Postulates 1 and 2, we can't even prove that a line contains any points. Clearly Postulates 1 and 2 are too weak to support the kind of theoretical structure we are trying to build. The text supplements them by adopting the powerful Ruler Postulate:

Postulate 3. (The Ruler Postulate.) The points of a line can be placed in correspondence with the real numbers in such a way that

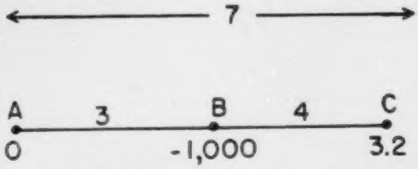
(1) To every point of the line there corresponds exactly one real number,

(2) To every real number there corresponds exactly one point of the line, and

(3) The distance between two points is the absolute value of the difference of the corresponding numbers.

This guarantees at one swoop that a line has the intrinsic properties we expect of it. Now the lines in every model of our theory will be well-behaved and richly endowed with points. It implies the congruence and order properties of a line in the conventional theory. Specifically it yields: (1) a form of the Location Property (Theorem 2-4); (2) that a segment can be "divided" into a given number of congruent "parts" - in particular it can be bisected (Theorem 2-5). It implies important order properties: Theorem 2-1 which says in effect that the order of points on a line in terms of geometric betweenness corresponds exactly to the order of their coordinates in terms of *algebraic* betweenness; and the Line Separation Property which is not explicitly dealt with in the text (see Commentary for Teachers, Chapter 2; also Problem 12 of Problem Set 3-3).

Observe the attractive inter-dependence of the weak Distance Postulate and the powerful Ruler Postulate. The first asserts the existence of a distance function but permits it to be completely trivial - the second tailors the line to our expectations but is impossible of statement without the notion of distance postulated in the first. Note that if we weaken the Ruler Postulate by dropping condition (3) and require merely the existence of a 1-1 correspondence between the points of a line and the set of real numbers, we may have pathological situations of the type indicated in the diagram. Here B is between A and C since AB + BC = AC, but -1,000, the coordinate of B, definitely is not between the coordinates of A and C.



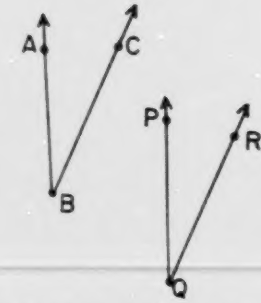
Our discussion suggests an important point in mathematical or deductive thinking. The Distance Postulate enables us to *define* betweenness but not to prove the *existence* of a single point between two given points. This is illustrated by the finite model above. The Ruler Postulate, however, implies the existence of infinitely many points between any two. This illustrates the point that a mathematical definition does not assert the *existence* of the entity defined. You may characterize the pot of gold at the end of the rainbow with great precision but you may experience equally great disappointment if you start to search for it before proving an existence theorem.

A final word. We may have oversold the deductive power of the Ruler Postulate and given you the impression that Postulates 1, 2 and 3 are sufficient for a complete theory of congruence. This is not so. Our theory so far is sufficient for the "linear" theory of congruence, specifically for congruence of segments - but not for congruence of more general figures like angles, triangles, circular arcs or triangular pyramids. For this we must introduce further postulates concerning congruence of angles and triangles. We discuss this in the next part since our main object here has been to indicate the flavor of the treatment in the text in contrast with the conventional one.

III. Congruence for Arbitrary Figures and Rigid Motions.

In this part we continue the discussion of congruence by indicating how it is successively defined for familiar elementary figures: angles, triangles, etc. Then using the simple and powerful modern idea of transformation we formulate the congruence concept for arbitrary figures - this surpasses in elegance and generality anything obtained in the field by the classical geometers. As a by-product we obtain - after two millenia - a precise mathematical concept of rigid motion. This is a great cultural achievement of our time. Rescuing from the jungles of physical intuition Euclid's crude superposition argument, we refine and perfect it to yield an objectively formulated concept which will be of use to human beings as long as they are impelled to think precisely about space.

III-1. Congruence of angles. The conventional treatment of angle congruence is similar to that sketched in Part I for congruence of segments - but naturally it is a bit more complicated since angles are more complex figures than segments. It begins with an undefined relation $\angle ABC \cong \angle PQR$ between two angles which as usual indicates that they are replicas of each other. This may be interpreted concretely to mean that if a frame composed of two jointed rods is set so that the rods coincide with the rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$, then without changing the setting the rods can be made to coincide with $\overrightarrow{QP}$ and $\overrightarrow{QR}$. We assume as for segments that congruence of angles is an equivalence relation:

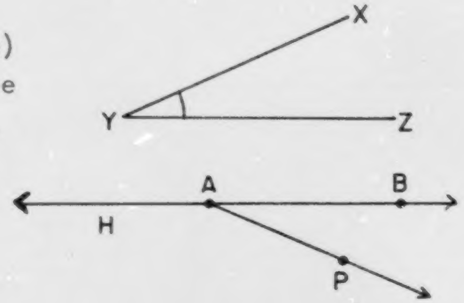


(1) (Reflexive Law) $\angle ABC \cong \angle ABC$;

(2) (Symmetry Law) If $\angle ABC \cong \angle PQR$ then $\angle PQR \cong \angle ABC$;

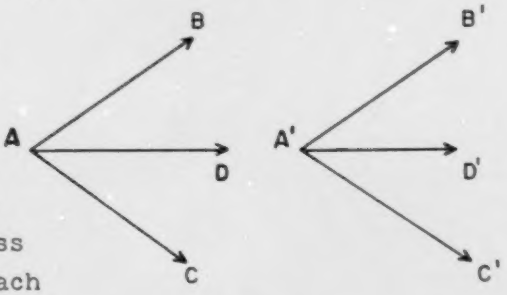
(3) (Transitive Law) If $\angle ABC \cong \angle PQR$ and $\angle PQR \cong \angle XYZ$ then $\angle APC \cong \angle XYZ$.

The Location Postulate for segments ((4), Section I-5) has the analogue



(4) (Angle Location Postulate) Let $\angle XYZ$ be any angle and $\overrightarrow{AB}$ be a ray on the edge of half-plane H. Then there is exactly one ray $\overrightarrow{AP}$, with P in H, such that $\angle PAB \cong \angle XYZ$.

And the Additivity Postulate ((5), Section I-5) appears in the form



(5) (Angle-Additivity Postulate) Suppose $\angle BAD \cong \angle B'A'D'$, $\angle DAC \cong \angle D'A'C'$, D is in the interior of $\angle BAC$ and D' is in the interior of $\angle B'A'C'$. Then $\angle BAC \cong \angle B'A'C'$.

Essentially on the basis of these postulates a measure process can be set up which assigns to each angle a unique positive real number called its measure in such a way that a fixed preassigned angle ("unit" angle) has measure 1 (compare Sections I-6 to I-9).

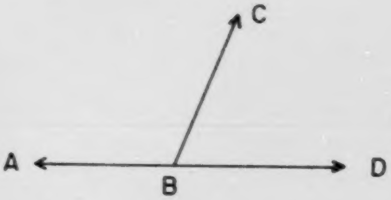
Denoting the measure of $\angle XYZ$ by $m\angle XYZ$, we have as you would expect from our discussion of measure of segments:

(1) $m\angle ABC = m\angle A'B'C'$ if and only if $\angle ABC \cong \angle A'B'C'$;

(2) If C is interior to $\angle ABD$ then $m\angle ABC + m\angle CBD = m\angle ABD$.

(Compare (1), (2) Section I-9).

But there are two properties which are unique to angular measure. First there is a real number b which is a least upper bound for the measure S of all angles (b is 180 in the familiar "degree measure"). Second the measure S of "supplementary adjacent" angles (i.e., a linear pair) always have a constant sum and this sum is the least upper bound b. Stated precisely: If $\angle ABC$ and $\angle CBD$ are a linear pair, then $m\angle ABC + m\angle CBD = b$.



III-2. Congruence of angles based on angular measure. We saw in (1) above that the conventional theory of angle congruence yields (as for segments) that two angles are congruent if and only if they have equal measures. This suggests (as for segments) that we assume the existence of angular measure and define congruence of angles in terms of it. Thus the treatment in the text assumes

Postulate 11. (The Angle Measurement Postulate.) To every angle $\angle ABC$ there corresponds a real number between 0 and 180, called the measure of the angle, and written as $m\angle ABC$, (compare the Distance Postulate).

Clearly our postulate has been set up so that the unit angle is the degree. In other words the angle characterized by $m\angle ABC = 1$ is what is usually defined to be a degree and will have the property that ninety such angles laid "side by side" will form a right angle. Precisely speaking the measure of a right angle will turn out to be 90. Notice that the measure of no angle can be 0 or 180 since our definition of angle restricts the side S to be non-collinear. (For a discussion of this restriction see Commentary for Teachers, Chapter 4.)

Now following a familiar path (Section II-5) we adopt the

Definition. $\angle ABC \cong \angle PQR$ means that $m\angle ABC = m\angle PQR$. Then properties (1), (2), (3) of III-1 above reduce to familiar equality properties of real numbers. The Angle Location Property ((4) above) must be postulated and is introduced in the form:

Postulate 12. (The Angle Construction Postulate.) Let $\overrightarrow{AB}$ be a ray on the edge of half-plane H. For every number r between 0 and 180 there is exactly one ray $\overrightarrow{AP}$, with P in H, such that $m\angle PAB = r$.

It might be thought now that the additivity property for angles ((5) above) could be derived as a theorem as was the corresponding property for segments (see (5), Section II-6). This isn't so. But it is a simple and important property of angles, and it is perfectly natural to postulate it:

Postulate 13. (The Angle-Addition Postulate.) If D is a point in the interior of $\angle BAC$, then $m\angle BAC = m\angle BAD + m\angle DAC$.

Finally we need a postulate to express the peculiarly "angular" property of supplementation:

Postulate 14. (The Supplement Postulate.) If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays and $\overrightarrow{AD}$ is another ray, then $m\angle DAC + m\angle DAB = 180$.

III-3. Congruence of triangles. We are now ready to consider congruence of triangles. Our definition of congruent triangles (Chapter 5 of text) is essentially the conventional one: One triangle is a "copy" of the other in the sense that its parts are "copies" of the corresponding parts of the other. But observe the precision with which it is formulated. The correspondence doesn't depend on individual interpretation of the vague term "corresponding" but is based objectively on a pairing of the vertices

$$A \longleftrightarrow A', \quad B \longleftrightarrow B', \quad C \longleftrightarrow C'$$

which induces a pairing of sides and of angles

$$\overline{AB} \longleftrightarrow \overline{A'B'}, \quad \overline{BC} \longleftrightarrow \overline{B'C'}, \quad \overline{CA} \longleftrightarrow \overline{C'A'}$$

$$\angle A \longleftrightarrow \angle A', \quad \angle B \longleftrightarrow \angle B', \quad \angle C \longleftrightarrow \angle C'.$$

Notice how spelling out the notion "corresponding" in this way helps to point up the importance of the notion of a congruence which is not mentioned in the conventional treatment. Thus our treatment brings to the fore the idea of a 1-1 correspondence between the vertices of Δ ABC and Δ A'B'C' which ensures that they are congruent because it requires corresponding sides and corresponding angles to be congruent, that is to have equal measures. This simple idea is capable of broad generalization.

Do we need postulates on congruence of triangles? We have a lot of information on congruence of segments and congruence of angles, separately - but nothing to inter-relate these ideas. For example, we can't yet prove the base angles of an isosceles triangle are congruent. Thus we introduce the S.A.S. Postulate to bind together our knowledge of segment congruence and angle congruence.

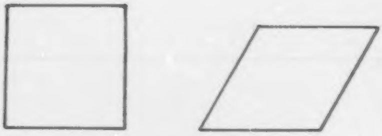
Now let us examine more closely the notion of congruence of triangles. Is it really necessary to require equality of measure of six pairs of corresponding parts? If we think of the sides of a triangle as its basic determining parts it seems very natural to define congruent triangles as having corresponding sides which are congruent. Naturally if we were to adopt this definition we would postulate that if the corresponding sides of two triangles are congruent their corresponding angles also are congruent, in order to ensure that this definition of congruent triangles is equivalent to the familiar one. Notice how much simpler the definition of a congruence between triangles becomes if we adopt the suggested definition. It is merely a 1-1 correspondence between the vertices of the triangles,

$$A \longleftrightarrow A', \quad B \longleftrightarrow B', \quad C \longleftrightarrow C'$$

which "preserves" distances in the sense that the distance between any two vertices of one triangle equals the distance between their corresponding vertices in the second triangle, that is

$$AB = A'B', \quad BC = B'C', \quad AC = A'C'.$$

III-4. *Congruence of quadrilaterals*. The main objection to the suggested definition is that it doesn't generalize in the obvious way for polygons - not even for quadrilaterals. This is attested by the fact that a square and a rhombus can have sides of the same length and not be congruent. So to guarantee congruence of quadrilaterals it is not sufficient to require just that corresponding sides be congruent, and it is customary to supplement this by requiring the congruence of corresponding angles. Thus the conventional definition requiring congruence both of sides and of angles applies equally well to triangles and quadrilaterals.



However angles, though very important, are rather strange creatures compared to segments and it seems desirable, if possible, to characterize congruent quadrilaterals in terms of congruent segments, or equivalently, equal distances. This is not so hard.

Going back to a triangle we observe that its three vertices taken two at a time yield three segments or three distances and that the figure is in a sense determined by these three distances. Similarly the four vertices of a quadrilateral yield not four, but six segments (the sides and the diagonals) and six corresponding distances, which serve to determine the quadrilateral. This suggests: If we have a 1-1 correspondence

$$A \longleftrightarrow A', \quad B \longleftrightarrow B', \quad C \longleftrightarrow C', \quad D \longleftrightarrow D'$$

between the vertices of the quadrilaterals $ABCD$, $A'B'C'D'$ such that corresponding distances are preserved, that is

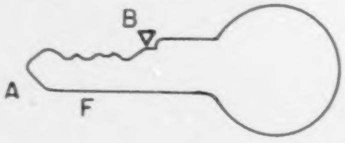
$$AB, AC, AD, BC, BD, CD = A'B', A'C', A'D', B'C', B'D', C'D'$$

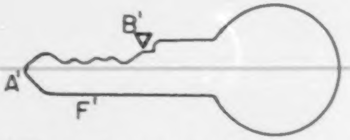
we call the correspondence a *congruence* and we write $ABCD \cong A'B'C'D'$. It is not hard to show this definition equivalent to the more familiar one.

III-5. Congruence of arbitrary figures. We now must face the problem of formulating a general definition of congruence. The piecemeal process we have employed, defining congruence separately for segments, angles, triangles, quadrilaterals is unavoidable in an elementary treatment but is neither satisfying nor complete. For it still remains to define congruent circles and congruent circular arcs and congruent ellipses and congruent rectangular solids, etc. In each case we construct an appropriate definition, we are sure it is correct, and are equally sure the general concept has eluded us.

So let's make a fresh start. Suppose F and F' are two congruent figures. Our basic intuition is that F' is an exact copy of F. Somehow this entails that each "part" of F' copies a corresponding "part" of F - that each point of F' behaves like some corresponding point of F. If F has a sharp point at A then F' must have a sharp point at a corresponding point A'; if F has maximum flatness at B then F' has maximum flatness at a corresponding point B'; if F has a largest chord $\overline{PQ}$ of length 12.3 then F' has a corresponding largest chord $\overline{P'Q'}$ of the same length, 12.3; and so on. How can we tie together these illustrations in a simple and precise way?

III-6. A congruence machine. Suppose instead of conceiving F' as a given copy of F, we take F and try to make a copy F' of it. As an illustration let F be a house key. Then F' can be produced by a key duplicating machine. The machine has the secret of the congruence concept - how does it work?





The machine has two moving parts: a scanning bar which traces the given key and a cutting bar which cuts a blank into a duplicate. As the scanning bar traces F starting at its tip A, the cutting bar traces the blank starting at its corresponding tip A'. As the scanner moves to position B, the cutter cuts away the metal and comes to rest at a corresponding position B'. When B rises to a "peak" so does

B' - when B falls to a trough so does B' - when B traverses a line segment, B' traverses a line segment of equal length.

What guarantees that this process yields a true copy? Simply this: When the scanner is fixed in a position B, the cutter comes to rest in a position B' such that distances AB and A'B' are equal. And this is true for each position B of the scanner. Clearly what the machine does is to associate to each chord $\overline{AB}$ from A of F an "equal" chord $\overline{A'B'}$ from A' of F'. And it associates the chords by associating their endpoints B and B'. Precisely speaking, the machine effects a 1-1 correspondence $X \longleftrightarrow X'$ between F and F' such that the distance AX always equals the distance A'X'.

Does this property hold just for A, the tip of F, and A' its correspondent in F'? Clearly not. The machine doesn't know where we start. What we have asserted about the chords of F from A will hold just as well for the chords from any point of F. So the 1-1 correspondence $X \longleftrightarrow X'$ between F and F' has the stronger property that for every choice of P and Q if $P \longleftrightarrow P'$, $Q \longleftrightarrow Q'$ then $PQ = P'Q'$, or as we say the correspondence preserves distance. Here we have the essence of the concept of congruence.

The legend has it that when Pythagoras succeeded in proving the theorem ascribed to him, he was so elated that he sacrificed a hecatomb of oxen to the gods. Surely in the light of this tradition the formal definition of congruence deserves a section all to itself.

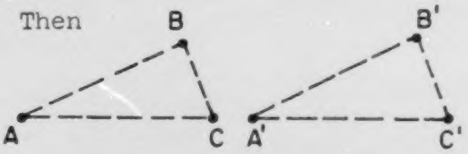
III-7. The definition. Let $X \longleftrightarrow X'$ be a 1-1 correspondence between two sets of points F, F' such that

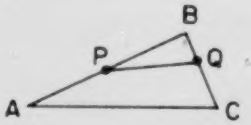
$$P \longleftrightarrow P', \quad Q \longleftrightarrow Q'$$

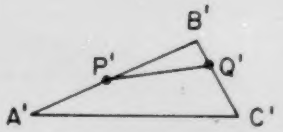
always implies $PQ = P'Q'$. Then we say F is congruent to F' and we write $F \cong F'$. Moreover we call the 1-1 correspondence a congruence between F and F'.

This definition is the culmination of two thousand years of thinking about congruence. Although it may seem quite abstract it unifies and unites the piecemeal discussion of congruence we have given. Every instance of congruent figures discussed above from segments to quadrilaterals can be proved to be a case of our general definition. This is discussed in detail in Appendix VIII of the text on Rigid Motion.

As a simple illustration of the definition let F and F' each be a triple of non-collinear points, say F is {A, B, C} and F' is {A', B', C'}. Let the 1-1 correspondence between F and F' which preserves distance be
(1) $A \longleftrightarrow A'$, $B \longleftrightarrow B'$, $C \longleftrightarrow C'$. Then we have AB = A'B', BC = B'C', AC = A'C'. We see intuitively that F' is a copy of F. Now shift from the point triples to the triangles they determine. The S.S.S. Theorem tells that Δ ABC is congruent to Δ A'B'C' in the conventional sense.



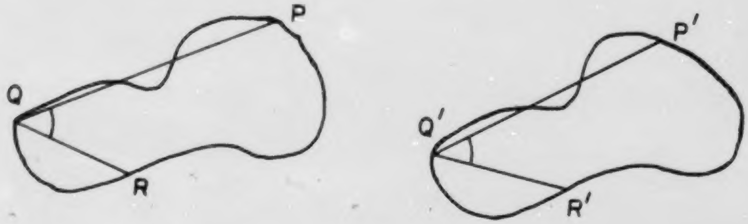




It follows (see Appendix VIII) that Δ ABC ≅ Δ A'B'C' in the sense of our definition. Actually there is a 1-1 correspondence between the infinite point sets Δ ABC, Δ A'B'C' which makes the vertices correspond as in (1) and which has the property that $P \longleftrightarrow P'$, $Q \longleftrightarrow Q'$ always implies PQ = P'Q'.

Observe how the correspondence between the triangle is engendered by the trivial seeming correspondence between their vertices. For example, if P is on $\overline{AB}$ its correspondent P' is determined as the unique point P' on $\overline{A'B'}$ such that A'P' = AP. Let us think of the finite set of its vertices, {A, B, C}, as a "skeleton" of Δ ABC. Then if the skeletons {A, B, C}, {A', B', C'} of two triangles are congruent the triangles as a whole are congruent - using "congruent" in its present sense. This idea was too complex to introduce in Chapter 5 of the text. But it was fore-shadowed there in the insistence that congruence of triangles was the consequence of the existence of a "congruence" between them - that is, a 1-1 correspondence between their sets of vertices which preserves lengths of sides and measures of angles.

There is an essential element of complexity in the definition of congruence: It requires (in general) the pairing off of the points of two infinite sets so as to preserve distance. This is unavoidable - it even seems to be present in the comparatively simple problem of duplicating keys. There is however an important element of simplicity: We don't have to mention angles and the preservation of their measures - the distance concept covers the situation. It follows easily that angle measures are preserved:



for if P⟷P', Q⟷Q', R⟷R' correspond under a congruence between F and F', and P, Q, R are non-collinear, we see by the S.S.S. Theorem that m∠ PQR = m∠ P'Q'R'.

You may find it interesting to give for quadrilaterals a discussion like the above for triangles - consider the vertex sets {A, B, C, D}, {A', B', C', D'} of quadrilaterals ABCD, A'B'C'D' as their "skeletons". In this connection recall the discussion of congruence of quadrilaterals at the end of Section III-4.

III-8. Motion in geometry. We can state the definition of rigid motion now, but it probably will be more meaningful if we say a few words first about the sense in which "motion" is used in contemporary geometry.

Let a body B move physically from an initial position F in space to a final position F'. It is not necessary for our purposes in geometry (as compared say with kinematics or fluid dynamics) to bother about the intermediate stages of the motion. So we can describe the motion merely by specifying the initial position X in F of an arbitrary point P of body B and its corresponding final position X' in F'. In its most general form, then, a motion is conceived as a 1-1 correspondence or transformation between two figures F and F'. The technical term "transformation" is often preferable to "motion" since it doesn't suggest various irrelevant attributes of physical motion.

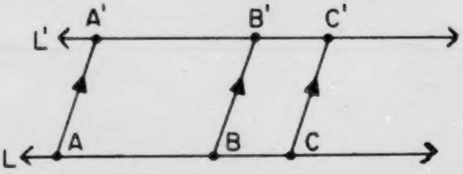
III-9. Rigid Motion. A motion or transformation between two point sets F and F' is a rigid motion if it preserves distances - that is if it is a congruence between F and F' as defined in Section III-7. A detailed discussion of the concept of rigid motion appears in Appendix VIII of the text.

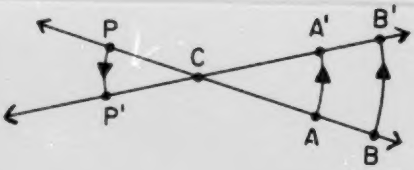
To introduce you to the modern theory of congruent figures and rigid motion we have put the main emphasis on the first, since it is more familiar and seems easier to apprehend. However, glancing back at the definition of congruent figures, you see it implicitly involves the notion of rigid motion. In fact now we can reword it: F is congruent to F' provided there exists a rigid motion between them, or as we say more graphically, a rigid motion which "transforms F into F'". This is the highly refined culmination of the vague and famous classical statement which served to introduce our discussion of congruence: "Two figures are congruent if they can be made to coincide by a rigid motion."

Sometimes the clarification of the basic concepts of a branch of mathematics firms up the foundations, puts the capstone on the superstructure and sets it to rest. This is not so here. The concept of rigid motion has stimulated the study of classical geometry, has yielded new insights and helped to unfold new unities. It has suggested the study of more general geometric transformations ("non-rigid motions") and has presented problems to the field of Modern Algebra, since motions tend to occur in certain "natural algebraic formations" called groups.

In the first place congruence and rigid motion have an impact on geometry since they apply to all figures. We can talk precisely not merely about congruence of (or rigid motion between) triangular pyramids or spherical zones or hyperbolic paraboloids but also of lines, planes, space, half-planes, rays, etc. At first it may sound silly to say a line is congruent to a line - but try to find a better replica of a line than a line! It must be just because the relation congruence applied to lines is so fundamental and universal that we are not conscious of it - as a fish must be unconscious of the notion humidity. In a first approach, congruence takes on importance as applied to segments (or angles or triangles) precisely because not all segments (or angles or triangles) are congruent to each other.

So it may seem trivial to say a line is congruent to a line or a plane to a plane or space to itself. But suppose we shift the focus from the static idea of congruent figures to the dynamic - and logically prior - idea of rigid motion. Is it trivial to say there exist rigid motions between lines or between planes or between space and itself? Just to ask this question discloses a broad vista: One of the principal concerns of contemporary geometry (or contemporary mathematics) is the study of transformations (rigid and non-rigid) of n-dimensional spaces.

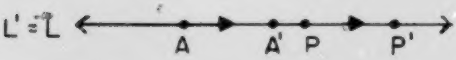


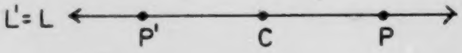


Consider the simplest case: Rigid motions which transform a line L into a line L'. If L ∥ L' we have slides or translations which "move" the points of L along parallel transversals to get their corresponding points of L'. If L and L' meet in just one point C we have a rotation about C. If L and L' coincide, that is L = L', we have two types of rigid motions operating on L:

(1) translations along L;

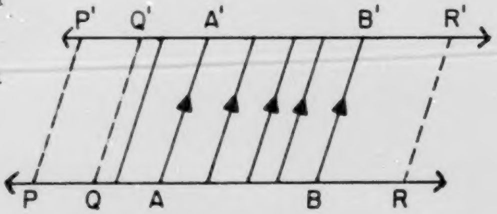
(2) reflections of L in a point C, where point C of L is "fixed" (that is it corresponds to itself) and every other point of L "moves" on L from one side of C to the other.





Similar considerations apply to planes. The theory culminates in the study of rigid motions of space - that is between space and itself. Here the basic types are translations, in which no point is fixed, rotations in which each point of a line (the axis of the rotation) is fixed, and reflections in a plane E in which each point of plane E is fixed and the half-spaces separated by E are "interchanged". More precisely a reflection in E is a transformation $X \longleftrightarrow X'$ such that if X is in E then X' = X and if X is not in E then E is the perpendicular bisector of $\overline{XX'}$. All rigid motions of space are "combinations" of these three basic types, just as all positive integers other than 1 are combinations of primes.

You may say that the theory of rigid motions of lines, planes and space is attractive and relatively simple, but haven't we left out the annoying complexities involved in the study of specific congruent figures like segments, truncated triangular pyramids and cones with oval bases? Not at all! They are elegantly covered in the theory of rigid motions of the basic "linear manifolds": line, plane, space.

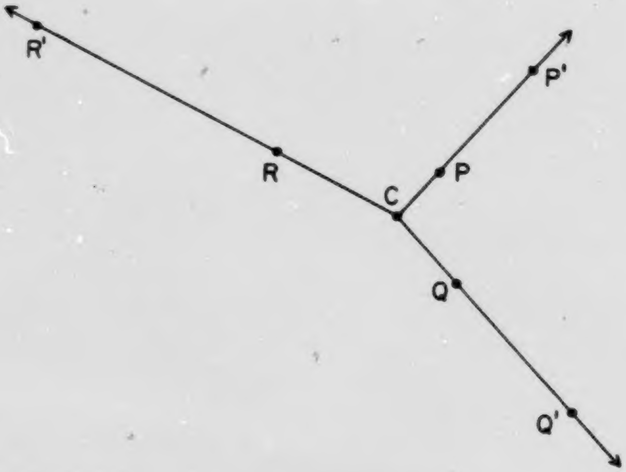


As a very simple illustration suppose segment $\overline{AB}$ is congruent to segment $\overline{A'B'}$. Then there is a rigid motion between them which, let us say, makes A correspond to A' and B to B'. Now we have the remarkable result that this rigid motion, which is a certain kind of 1-1 correspondence between segments $\overline{AB}$ and $\overline{A'B'}$ can be extended to form a rigid motion between the whole line $\overleftrightarrow{AB}$ and the whole line $\overleftrightarrow{A'B'}$ - and this extension can be made in just one way. Thus we don't disturb the correspondence between $\overline{AB}$ and $\overline{A'B'}$ but "amplify" it by suitably defining a unique correspondent for each point of $\overleftrightarrow{AB}$ not in $\overline{AB}$, so that the final correspondence is a rigid motion between $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$. So in the study of rigid motions between lines as wholes, we are automatically covering all possible rigid motions (and hence all possible relations of congruence) between "linear" figures; (that is, subsets of lines which contain more than one point). Similarly any rigid motion between "planar" figures (that is, subsets of a plane which are not contained in any line) is uniquely extendable to a rigid motion of their containing planes. Finally we observe that any conceivable rigid motion is encompassed by a rigid motion of space.

III-10. Non-rigid motions. As we have indicated, modern geometry is concerned with transformations that do not preserve distance, as well as with those which do. In Euclidean Geometry the most important example is a similarity, which bears the same relation to similar figures that a congruence or rigid motion does to figures which are congruent. Formally suppose $X \longleftrightarrow X'$ is a 1-1 correspondence between figures F and F' such that

$$P \longleftrightarrow P', \quad Q \longleftrightarrow Q'$$

always implies $P'Q' = k \cdot PQ$ where k is a fixed positive number. Then we call the correspondence a similarity transformation or a similarity and we say F is similar to F'. It easily follows that a similarity transformation - although it is not in general a rigid motion - always preserves angle measures. This definition of similar figures, when restricted to triangles, can be proved equivalent to the familiar one. The simplest general type of similarity is the dilatation (in a plane or in space) - this is a similarity which leaves a given point C fixed and radially "stretches" the distance of any point from C by a positive factor k.



Other important types of transformations are central in various geometric theories. For example, "parallel projection" between planes in affine geometry; "central projection" between planes in projective geometry; and topological transformations, which are a type of continuous 1-1 correspondence, in topology. The theory of map-making is concerned with various "projections" or other kinds of transformations between a sphere and a cone, cylinder or plane.

And so we have ended our talk by touching upon a modern generalization of rigid motion which well might merit a talk for itself.

INTRODUCTION TO NON-EUCLIDEAN GEOMETRY

About one hundred and fifty years ago, a revolution in mathematical thought began with the discovery of a geometrical theory which differed from the classical theory of space formulated by Euclid about 300 B.C. Euclid's Geometry Text, the Elements, was the finest example of deductive thinking the human race had known, and had been so considered for two thousand years. It was believed to be a perfectly accurate description of physical space, and at the same time, the only way in which the human mind could conceive space. It is no small wonder then that the development of theories of non-Euclidean geometry had an impact on mathematical thought comparable to that of Darwin in biology, Copernicus in astronomy or Einstein in physics.

How did this revolutionary change come about? Strangely enough it may be considered to have had its origin in Euclid's text. Although he lists his postulates at the beginning, he refrains from employing one of them until he can go no farther without it. This is the famous fifth postulate which we may state in equivalent form as

Euclid's Parallel Postulate. If point P is not on line L, there exists only one line through P which is parallel to L.

It seems probable that Euclid deferred the introduction of the fifth postulate because he considered it more complex and harder to grasp than his other postulates.

The consequences of introducing Euclid's Parallel Postulate are almost phenomenal. Using it we get in sequence:

1. The Alternate Interior Angle Theorem for parallel lines;

2. The sum of the measures of the angles of a triangle is 180;

3. Parallel lines are everywhere equidistant;

4. The existence of rectangles of preassigned dimensions.

As remote but recognizable consequences of Euclid's Parallel Postulate, we have:

5. The familiar theory of area in terms of square units which in effect reduces any plane figure to an equivalent rectangle;

6. The familiar theory of similarity;

7. The Pythagorean Theorem.

It is hard to see how any of these important results could be proved without recourse to Euclid's Parallel Postulate or an equivalent assumption.

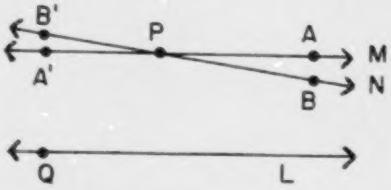
There is no explicit evidence that Euclid considered the fifth postulate an improper assumption in his basis for geometry. But generations of mathematicians for over 2000 years were dissatisfied with it, and worked hard and long in attempts to deduce it as a theorem from the other seemingly simpler postulates. Right up to the beginning of the 19th century able mathematicians convinced themselves that they had settled the problem only to have flaws discovered in their work. Sometimes they employed the principle of the indirect method and developed elaborate and subtle arguments to prove that the denial of Euclid's Parallel Postulate would force one into a contradiction. None of these arguments stood up under analysis. Finally early in the 19th century, J. Bolyai (1802-1860) a Hungarian army officer, and N. I. Lobachevsky (1793-1856) a Russian professor of mathematics at the University of Kazan, independently introduced theories of geometry based on a contradiction of Euclid's Parallel Postulate.

The purpose of this talk is to give an elementary introduction to the non-Euclidean theory of geometry which Bolyai and Lobachevsky created.

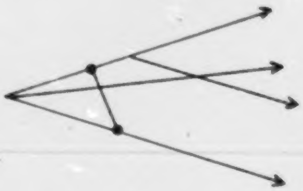
I. Two Non-Euclidean Theorems

In this part we try to give you - without a long preliminary discussion - the flavor of non-Euclidean geometry. Our viewpoint is this: Suppose we consider the hypothesis that there are two lines parallel to a particular line through a particular point. What will follow? As a basis for our deductions we assume the postulates of Euclidean geometry except the Parallel Postulate, specifically Postulates 1, ..., 15 of the text.

Theorem 1. Let P be a point and L a line such that there are two lines through P each of which is parallel to L. Then L is wholly contained in the interior of some angle.



Proof: Let lines M and N contain P and be parallel to L. Then M and N separate the plane into four "parts" each of which is the interior of an angle. Specifically these parts or regions may be labelled as the interiors of the angles $\angle$ APB, $\angle$ A'PB', $\angle$ A'PB, $\angle$ APB' where P is between A and A' on M and P is between B and B' on N. Let Q be any point of L. Since L does not meet M or N, Q is not on M or N. So Q is in one of the four angle interiors say the interior of $\angle$ A'PB. Now where can L lie? Note that one of its points Q is in the interior of $\angle$ A'PB and that L does not meet the sides of the angle $\angle$ A'PB. Clearly L is trapped inside $\angle$ A'PB and the theorem is proved.



Observe how strange this is when compared with the Euclidean situation where only a part of a line can be contained in the interior of an angle, as indicated in the figure. But note - as always in mathematics - the inevitability of the result once the hypothesis is granted. You may say the argument is valid abstractly - but it doesn't correspond to physical reality.

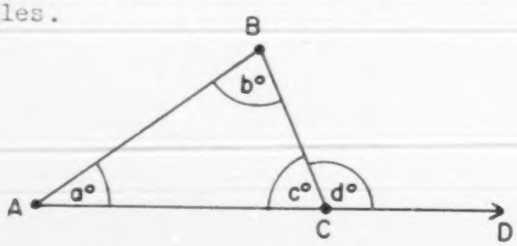
As you make a statement like this you begin to tread the path of the non-Euclidean geometers. All that one needs to think mathematically is a set of precisely stated assumptions (postulates) from which conclusions (theorems) can be derived by logical reasoning. Are these assumptions absolutely true when applied to the physical world? We don't really know. It is not our professional concern as mathematicians to answer the question. It lies in the domain of physicists, astronomers and surveyors. As human beings who work in mathematics we may like to feel that our theories are applicable to physical reality. But this doesn't require the absolute truth of our postulates or our theorems. When Euclidean geometry is applied by an architect or engineer or surveyor he doesn't require results which are absolutely correct - he might consider this a mirage. Rather he demands results correct to the degree of precision required by his problem - accuracy of one part in a hundred might be excellent in a pocket magnifying glass but one part in a million might be too rough for a far-ranging astronomical telescope.

Our first theorem indicated how positional or non-metrical properties in a non-Euclidean geometry might differ from our Euclidean expectations. Now we show how metrical properties - specifically the angle sum of a triangle - are altered when we change the Parallel Postulate.

Theorem 2. Let P be a point and L a line such that there are two lines through P each of which is parallel to L. Then there exists at least one triangle the sum of whose angle measures is less than 180.

We first prove a lemma.

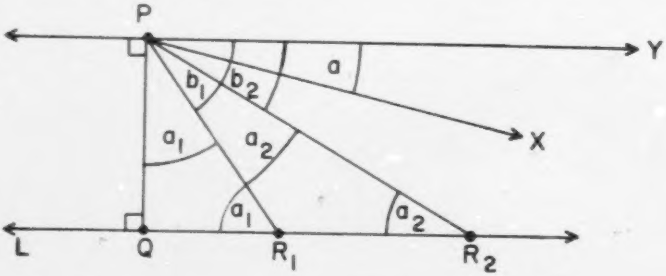
Lemma. If the sum of the angle measures of a triangle is greater than or equal to 180 then the measure of an exterior angle is less than or equal to the sum of the measures of the two remote interior angles.



Proof: We have $a + b + c \geq 180$. Hence

$$a + b \geq 180 - c = d.$$

Proof of Theorem 2: Suppose the theorem false. Then the sum of the angle measures of every triangle is greater than or equal to 180.



Let L be a line and P a point such that there are two lines through P parallel to L. Let line $\overleftrightarrow{PQ}$ be perpendicular to L at Q. Since there are two lines through P parallel to L one of these must make an acute angle with line $\overleftrightarrow{PQ}$. Suppose then line $\overleftrightarrow{PX}$ is parallel to L and makes an acute angle, $\angle QPX$, with line $\overleftrightarrow{PQ}$. Let line $\overleftrightarrow{PY}$ be perpendicular to line $\overleftrightarrow{PQ}$ with Y on the same side of line $\overleftrightarrow{PQ}$ as X. Let $m\angle YPX = a$; then $a < 90$. (Think of a as a small positive number, say .1.) Now locate R_1 on L so that $QR_1 = PQ$ and R_1 is on the same side of $\overleftrightarrow{PQ}$ as X and Y. Draw segment $\overline{PR_1}$. Then $\Delta\ PQR_1$ is isosceles so that $m\angle QPR_1 = m\angle QR_1P = a_1$. Since the exterior angle of $\Delta\ PQR_1$ at Q is a right angle, the Lemma implies

$$a_1 + a_1 = 2a_1 \geq 90$$

and

$$a_1 \geq 45.$$

Let $m\angle YPR_1 = b_1$. Then

$$b_1 + a_1 = 90,$$

so that $b_1 = 90 - a_1$.

and $b_1 \leq 45.$

Moreover $b_1 > a.$

Now we repeat the argument by constructing a new triangle. Extend segment $\overline{QR_1}$ to R_2 making $R_1R_2 = PR_1$. Draw $\overline{PR_2}$. Then $\Delta\ PR_1R_2$ is isosceles, so that $m\angle R_1PR_2 = m\angle R_1R_2P = a_2$. By the Lemma

$$a_2 + a_2 = 2a_2 \geq a_1.$$

So that $2a_2 \geq a_1 \geq 45$

and $a_2 \geq \frac{45}{2}.$

Let $m\angle YPR_2 = b_2$. Then

$$b_2 + a_2 = b_1,$$

$$b_2 = b_1 - a_2.$$

Since $b_1 \leq 45$ and $a_2 \geq \frac{45}{2}$ we have

$$b_2 \leq \frac{45}{2}.$$

Moreover $b_2 > a.$

Continuing in this way we obtain a sequence of real numbers

$$b_1,\ b_2,\ b_3,\ \ldots$$

which are less than or equal to respectively

$$45,\ \frac{45}{2},\ \frac{45}{4},\ \ldots$$

but all of which are greater than the fixed positive number a. This is impossible since repeated halving of 45 must eventually produce a number less than a. So our supposition is false and the theorem holds.

A proof of this type, though not difficult, may be unfamiliar and you may have to mull it over a bit to appreciate it better. In intuitive terms it is not very hard. There are two main points. First, the ray $\overrightarrow{PX}$ which doesn't meet L acts as a sort of boundary for the rays $\overrightarrow{PR_1}$, $\overrightarrow{PR_2}$, ... which do meet L. Thus the angles $\angle YPR_1$, $\angle YPR_2$, ... have measures b_1, b_2, ... which are greater than a. On the other hand (if the sum of the angle measures of every triangle is at least 180) we can pile up successive angles $\angle QPR_1$, $\angle R_1PR_2$, ..., starting at ray $\overrightarrow{PQ}$, of measures at *least* 45, $\frac{45}{2}$, $\frac{45}{4}$, ... so that the angles $\angle YPR_1$, $\angle YPR_2$, ... have measures at *most* 45, $\frac{45}{2}$, $\frac{45}{4}$, ... So we have a contradiction in that the angles $\angle YPR_1$, $\angle YPR_2$, ... have measures which approach zero but are all greater than a fixed positive number a.

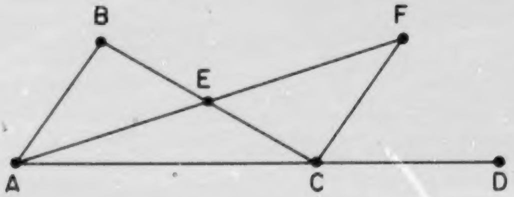
A final remark. You may object that we have not really justified that $\overrightarrow{PX}$ is a "boundary" for $\overrightarrow{PR_1}$, $\overrightarrow{PR_2}$, ... To take care of this observe that $\overrightarrow{PR_1}$ and $\overrightarrow{PX}$ are on the same side of line $\overleftrightarrow{PQ}$. Consequently one of them must fall inside the angle formed by $\overrightarrow{PQ}$ and the other. Suppose $\overrightarrow{PX}$ fell inside $\angle QPR_1$. Then $\overrightarrow{PX}$ would meet line $\overleftrightarrow{QR_1}$. Since this is impossible, $\overrightarrow{PR_1}$ must lie inside $\angle QPX$. Similarly for $\overrightarrow{PR_2}$, ...

II. Neutral Geometry

We are using the term "neutral geometry" in this part to indicate that we are assuming neither Euclid's Parallel Postulate nor its contradictory. We shall merely deduce consequences of Euclid's Postulates other than the Parallel Postulate, (specifically our discussions are based on Postulates 1, ..., 15 of the text). Our results then will hold in Euclidean Geometry and in the non-Euclidean geometry of Bolyai and Lobachevsky since they are deducible from postulates which are common to both theories. Our study is neutral also in the sense of avoiding controversy over the Parallel Postulate. Actually its study helps us to accept the idea of non-Euclidean geometry since it points up the fact that mathematically we have a more basic geometrical theory which can be definitized in either of two ways.

We proceed to derive some results in neutral geometry. Since you are familiar with so many striking and important theorems which do depend on Euclid's Parallel Postulate you might think that there are no interesting theorems in neutral geometry. However, this is not so. First we sketch the proof of a familiar and important theorem of Euclidean geometry whose proof does not depend on a parallel postulate (see text, Theorem 7-1).

Theorem 3. An exterior angle of a triangle is larger than either remote interior angle.



Proof: Given Δ ABC with exterior angle $\angle$ BCD. We show m$\angle$ BCD is greater than m$\angle$ B and m$\angle$ A. Let E be the mid-point of segment $\overline{BC}$ and let F be the point such that AE = EF and E is between A and F. It follows that Δ BEA $\cong$ Δ CEF so that

$$m\angle B = m\angle ECF.$$

But

$$m\angle BCD = m\angle ECF + m\angle FCD.$$

Substituting $m\angle B$ for $m\angle ECF$ we have

$$m\angle BCD = m\angle B + m\angle FCD,$$

so that

$$m\angle BCD > m\angle B.$$

The proof is completed as usual by applying the above argument to show that the vertical angle of $\angle BCD$ is larger than $\angle A$.

Corollary 1. The sum of the measures of two angles of a triangle is less than 180.

Proof: Given $\triangle ABC$ we show $m\angle A + m\angle B < 180$. By the theorem $m\angle A$ is less than the measure of an exterior angle at B. Thus

$$m\angle A < 180 - m\angle B$$

so that

$$m\angle A + m\angle B < 180.$$

This corollary is important since, without assuming a parallel postulate, it gives us information about the angles of a triangle. It tells us for example, that a triangle can have at most one obtuse angle or at most one right angle.

Corollary 2. In a plane two lines are parallel if they are both perpendicular to the same line (compare text, Theorem 9-2).

Proof: The basic properties of perpendicular lines in Euclidean geometry are studied prior to the introduction of the Parallel Postulate, and so are part of (or are valid in) neutral geometry. Thus the familiar proof of the corollary is applicable: If the two lines met we would have, in a plane, two lines perpendicular to the same line at the same point. This is impossible and the lines can't meet.

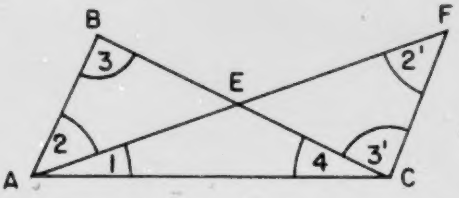
Corollary 3. Let L be a line, and let P be a point not on L. Then there is at least one line through P, parallel to L.

Proof: This follows from Corollary 2 by the familiar theorem on the existence of perpendiculars: Let $L_1 \perp L$ through P, and $L_2 \perp L_1$ through P. Then $L_2 \parallel L_1$.

Observe that this familiar - almost hackneyed - discussion has yielded a very important principle: That parallel lines exist. More precisely, there exists at least one line parallel to a given line through an external point. And we got this result without assuming any parallel postulate! So the crucial point in our study of the theory of parallelism will be whether there is one, or more than one, line parallel to a given line through an external point.

To prove an important, and not sufficiently well known, theorem of Legendre (1752-1833) we introduce the following:

Lemma. Given Δ ABC and $\angle$ A. Then there exists a triangle $\Delta A_1B_1C_1$ such that: (a) it has the same angle measure sum as Δ ABC; (b) $m\angle A_1 \leq \frac{1}{2}m\angle A$.



Proof: We use the same construction as in Theorem 3. Let E be the mid-point of $\overline{BC}$ and let F satisfy AE = EF and E is between A and F. Then Δ BEA $\cong$ Δ CEF and corresponding angles have equal measures. Δ AFC is the $\Delta A_1B_1C_1$ we are seeking. We have

$$\begin{aligned} m\angle A + m\angle B + m\angle C &= m\angle 1 + m\angle 2 + m\angle 3 + m\angle 4 \\ &= m\angle 1 + m\angle 2' + m\angle 3' + m\angle 4 \\ &= m\angle CAF + m\angle AFC + m\angle FCA. \end{aligned}$$

To complete the proof note that

$$m\angle A = m\angle 1 + m\angle 2 = m\angle 1 + m\angle 2'$$

so that

$$m\angle A = m\angle CAF + m\angle AFC.$$

Hence one of the terms on the right is less than or equal to $\frac{1}{2}$ the term on the left, that is $\frac{1}{2}m\angle$ A. Consequently Δ AFC can be relabeled $\Delta A_1B_1C_1$ so as to make the theorem valid.

Note that since we have not assumed Euclid's Parallel Postulate we don't know that the angle measure sum is constant for all triangles. So the lemma is a significant result in that we can construct from a given triangle a new one with the same angle measure sum. In intuitive terms we can replace a triangle by a "slenderer" one without altering its angle measure sum. In effect the proof shows this by cutting off Δ ABE from Δ ABC and pasting it back on as Δ FCE.

Now we can prove the following remarkable theorem.

Theorem 4. (Legendre.) The angle measure sum of any triangle is less than or equal to 180.

Proof: Suppose the contrary. Then there must exist a triangle, Δ ABC, whose angle measure sum is $180 + p$, where p is a positive number. Now we apply the Lemma. It tells us that there exists a slenderer triangle, $\Delta A_1B_1C_1$, whose angle measure sum also is $180 + p$ such that $m\angle A_1 \leq \frac{1}{2}m\angle A$.

To fix our ideas let us say $p = 1$ and $m\angle A = 25$. Then

$$m\angle A_1 + m\angle B_1 + m\angle C_1 = 181 \text{ and } m\angle A_1 \leq \frac{25}{2}.$$

Pressing our advantage we reapply the lemma. So there is a still slenderer triangle, let us call it $\Delta A_2B_2C_2$, whose angle measure is $180 + p$ and $m\angle A_2 \leq \frac{1}{2}m\angle A_1$. That is

$$m\angle A_2 + m\angle B_2 + m\angle C_2 = 181 \text{ and } m\angle A_2 \leq \frac{25}{4}.$$

Continuing in this way, we get a sequence of triangles each with angle measure sum 181 and with successive angles of measures no greater than

$$25, \frac{25}{2}, \frac{25}{4}, \frac{25}{8}, \ldots .$$

To see this is impossible, consider $\Delta A_5B_5C_5$ for which $m\angle A_5 < p$.

We have

$$m\angle A_5 + m\angle B_5 + m\angle C_5 = 181 \quad \text{and} \quad m\angle A_5 \leq \frac{25}{32}.$$

Certainly

$$m\angle A_5 < 1,$$

but

$$m\angle B_5 + m\angle C_5 < 180$$

by Corollary 1 to Theorem 3. Adding the inequalities,

$$m\angle A_5 + m\angle B_5 + m\angle C_5 < 181.$$

This contradiction implies our supposition false, and the theorem is established.

Note the point of the proof is to get a triangle so "slender", that is with one angle so small, that the triangle can't exist by Corollary 1 above. It may now be instructive to write out the proof in general terms without assigning specific values to p and $m\angle A$.

Corollary 4. The angle measure sum of any quadrilateral is less than or equal to 360.

III. Do Rectangles Exist?

We continue to study neutral geometry, and are interested in whether a rectangle can exist in such a geometry, and what happens if it does. So most of our theorems will have the hypothesis that a rectangle exists. We use freely the results of Part II on neutral geometry.

The existence of a rectangle in a geometry is not a trivial thing - imagine what Euclidean geometry would be like if you didn't have or couldn't use rectangles. If you try to construct a rectangle you will find you are assuming Euclid's Parallel Postulate or one of its consequences, such as, the angle measure sum of a triangle is 180.

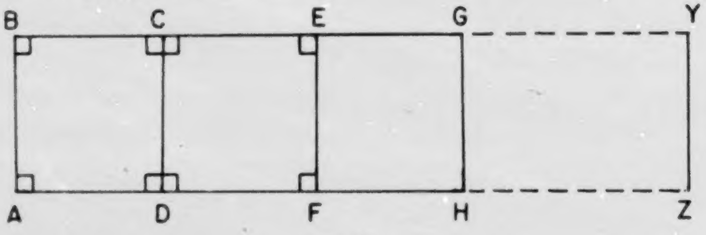
First, to avoid ambiguity, we formally define rectangle as we shall use the term:

Definition. A (plane) quadrilateral is called a rectangle if each of its angles is a right angle.

Notice that since we are operating in neutral geometry and have not assumed Euclid's Parallel Postulate, we can't automatically apply familiar Euclidean propositions, such as (1) the opposite sides of a rectangle are parallel, or (2) that they are equal in length, or (3) that a diagonal divides a rectangle into two congruent triangles. If we want to assert any of these results we will have to prove them from our definition without assuming a parallel postulate. For example, (1) is immediate by Corollary 2.

Theorem 5. If one particular rectangle exists then a rectangle exists with an arbitrarily large side.

Restatement: Suppose a rectangle ABCD exists and x is a given positive real number. Then there exists a rectangle with one side of length greater than x.



Proof: We use ABCD as a "building block" to construct the desired rectangle. Construct a quadrilateral DCEF congruent to ABCD, so that $\overline{EF}$ and $\overline{AB}$ are on opposite sides of line $\overleftrightarrow{CD}$. Then DCER is a rectangle. Moreover, B, C, E lie on a line by a familiar perpendicularity property. Similarly, A, D, F are collinear. So ABCEFD is a quadrilateral ABEF and consequently a rectangle. Note ABEF has the property that

$$AF = 2AD.$$

Similarly we construct FEGH a congruent replica of ABCD so that $\overline{GH}$ and $\overline{AB}$ are on opposite sides of line $\overleftrightarrow{EF}$. And we see that ABGH is a rectangle such that

$$AH = 3AD.$$

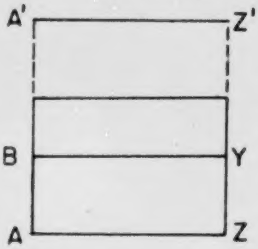
Continuing in this way we can construct a rectangle ABYZ such that

$$AZ = nAD$$

for each positive integer n. Now choose n so big that $nAD > x$. Then ABYZ satisfies the conditions of our theorem.

Corollary 5. If one particular rectangle exists, then a rectangle exists with two arbitrarily large adjacent sides.

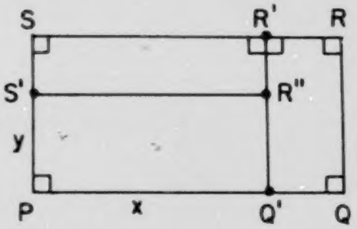
Restatement: Suppose a rectangle ABCD exists and x, y are given positive real numbers. Then there exists a rectangle PQRS such that $PQ > x$ and $PS > y$.



Proof: By the theorem we have a rectangle ABYZ with $AZ > x$. By placing successive congruent replicas of ABYZ "on top" of each other starting with ABYZ, we eventually get a rectangle AA'Z'Z with $AA' > y$ and $AZ > x$.

Theorem 6. If one particular rectangle exists then a rectangle exists with two adjacent sides of preassigned lengths x, y.

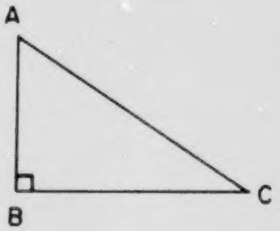
Proof: Our method is that of a tailor: By the last corollary we get a rectangle $PQRS$ such that $PQ > x$ and $PS > y$; then we cut it down to fit.

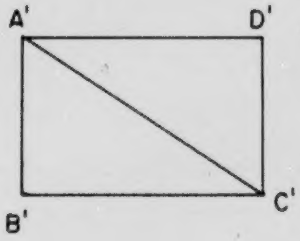


There is a point Q' in $\overline{PQ}$ such that $PQ' = x$. Drop a perpendicular from Q' to line $\overleftrightarrow{RS}$ with foot R'. We show $PQ'R'S$ is a rectangle. It certainly has right angles at P, S, R'. We show $\angle PQ'R'$ also is a right angle. Suppose $m\angle PQ'R' > 90$. Then the sum of the angle measures of quadrilateral $PQ'R'S$ is greater than 360 contrary to the corollary of Legendre's Theorem (Part II). Suppose $m\angle PQ'R' < 90$. Then $m\angle QQ'R' > 90$ and quadrilateral $QQ'R'R$ has an angle measure sum greater than 360. Thus the only possibility is $m\angle PQ'R' = 90$, and $PQ'R'S$ is a rectangle.

In the same way there is a point S' in $\overline{PS}$ such that $PS' = y$. Drop a perpendicular from S' to line $\overleftrightarrow{Q'R'}$ with foot R''. Then as above $PQ'R''S'$ is a rectangle, and it has sides $\overline{PQ'}$ and $\overline{PS'}$ of lengths x and y.

Theorem 7. If one particular rectangle exists then every right triangle has an angle measure sum of 180.



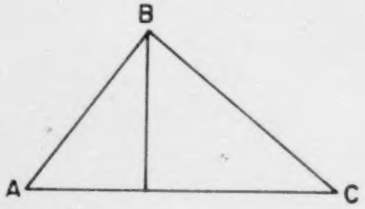


Proof: Our procedure is to show: (1) any right triangle is congruent to a triangle formed by the splitting of a rectangle by a diagonal, and (2) the latter type of triangle must have an angle measure of 180. Let Δ ABC be a right triangle with right angle at B. By Theorem 6 there exists a rectangle A'B'C'D' with A'B' = AB and B'C' = BC. Draw $\overline{A'C'}$. Then Δ ABC $\cong \Delta$ A'B'C' and they have the same angle measure sum. Let p be the angle measure sum of Δ A'B'C' and q be that of Δ A'C'D'. We have

$$p + q = 4 \cdot 90 = 360. \qquad (1)$$

We want to show $p = 180$. By Legendre's Theorem $p < 180$ or $p = 180$. Suppose $p < 180$. Then by (1) $q > 180$, contrary to Legendre's Theorem. So $p = 180$ must hold and the proof is complete.

Theorem 8. If one particular rectangle exists then every triangle has an angle measure sum of 180.



Proof: Any triangle Δ ABC can be split into two right triangles. Each of these has angle measure sum 180 by Theorem 7. It easily follows that the same holds for Δ ABC.

This is a rather striking result: The existence of one puny rectangle with microscopic sides inhabiting a remote portion of space guarantees that every conceivable triangle has an angle measure sum of 180. Since this is a typically Euclidean Property we are tempted to say that if in a neutral geometry a rectangle exists, the geometry must be Euclidean. The statement is correct but not fully justified, since to characterize a neutral geometry as Euclidean we must know that it satisfies Euclid's Parallel Postulate. This can now be proved without trouble.

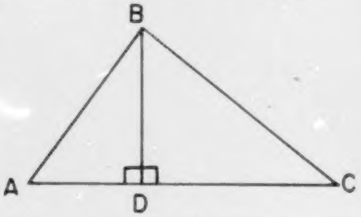
Theorem 9. If one particular rectangle exists then Euclid's Parallel Postulate holds.

Proof: Suppose a rectangle exists but Euclid's Parallel Postulate fails. Then there must exist a line L and a point P such that there are two lines through P parallel to L, since by Corollary 3 there is at least one line parallel to a given line through an external point. Then by Theorem 2 there exists one triangle, at least, whose angle measure sum is less than 180. This contradicts Theorem 8. Consequently Euclid's Parallel Postulate must hold.

What we have justified is a remarkable equivalence theorem, namely: Euclid's Parallel Postulate is logically equivalent to the existence of a rectangle. That is, taking either of these statements as a postulate we can deduce the other as a theorem, provided of course we assume the postulates for a neutral geometry.

An interesting condition equivalent to the existence of a rectangle is the existence of a triangle whose angle measure is 180:

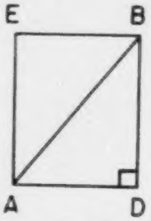
Theorem 10. If there exists one particular triangle with angle measure sum of 180, then there exists a rectangle.



Proof: Suppose Δ ABC has angle measure sum 180. First we show there is a right triangle with angle measure sum 180. Split Δ ABC into two right triangles, whose angle measure sums are say p and q. Then

$$p + q = 180 = 2 \cdot 90 = 360.$$

We show $p = 180$. By Legendre's Theorem, $p \leq 180$. If $p < 180$ then $q > 180$ contrary to Legendre's Theorem. Thus there is a right triangle, say Δ ABD, which has angle measure sum 180.



Now we put two such right triangles together to form a rectangle. Construct $\Delta AEB \cong \Delta BDA$ with E on the opposite side of line $\overleftrightarrow{AB}$ from D. Show ADBE is a rectangle.

Corollary 6. If one particular triangle has angle measure sum 180 then every triangle has angle measure sum 180.

Proof: By Theorems 10 and 8.

Corollary 7. If one particular triangle has angle measure sum 180 then Euclid's Parallel Postulate holds.

Proof: By Theorems 10 and 9.

Corollary 8. If one particular triangle has an angle measure sum which is less than 180 then every triangle has an angle measure sum less than 180.

Proof: Suppose Δ ABC has angle measure sum less than 180. Consider any triangle Δ PQR. By Legendre's Theorem its angle measure sum p must satisfy $p = 180$ or $p < 180$. Suppose $p = 180$. Then by Corollary 6, Δ ABC has angle measure sum 180, contrary to hypothesis. Thus $p < 180$.

Comparing Corollaries 6 and 8 we observe an important fact. A neutral geometry is "homogeneous" in the sense that all of its triangles have an angle measure sum of 180 or they all have angle measure sums less than 180. The first type of neutral geometry is merely Euclidean geometry - the second type corresponds to the non-Euclidean geometry developed by Bolyai and Lobachevsky. This will be discussed in the next part.

Exercise 1. Suppose there is only one line parallel to a particular line L through a particular point P. Prove that Euclid's Parallel Postulate holds.

Exercise 2. Suppose there are two lines parallel to a particular line L through a particular point P. Prove there are two lines parallel to each line through each external point.

IV. Lobachevskian Geometry

Now we introduce the non-Euclidean geometry of Bolyai and Lobachevsky as a formal theory based on its own postulates. We call the theory Lobachevskian geometry to signalize the lifetime of work which Lobachevsky devoted to the theory. To study Lobachevskian geometry we merely assume the postulates of Euclidean geometry but replace Euclid's Parallel Postulate by Lobachevsky's Parallel Postulate: If point P is not on line L there are at least two lines through P which are parallel to L. In other words we assume the postulates of neutral geometry (Postulates 1, ..., 15 of the text) and adjoin Lobachevsky's Parallel Postulate. Consequently the theorems which we have already derived are valid in Lobachevskian geometry. In fact, by putting together two earlier results we get the following important theorem.

Theorem 11. The angle measure sum of any triangle is less than 180.

Proof: By Theorem 2 there exists a triangle whose angle measure sum is less than 180. Hence the same is true of every triangle by Corollary 8.

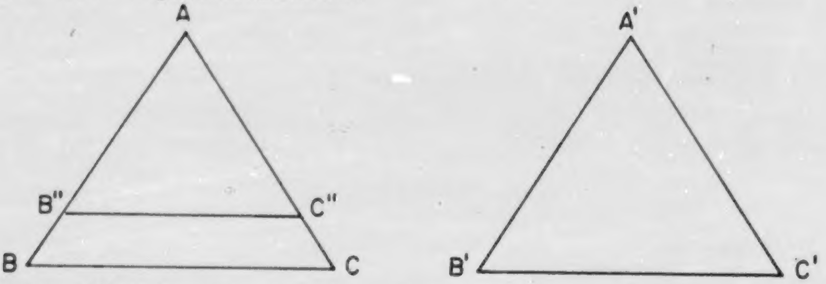
Corollary 9. The angle measure sum of any quadrilateral is less than 360.

Proof: By the corollary to Legendre's Theorem (Part II, Theorem 2) the only other possibility for the value is 360 - and this is ruled out by Theorem 11.

Corollary 10. There exist no rectangles.

Now we show that similar triangles can't exist in Lobachevskian geometry, except of course for the trivial case of congruent triangles.

Theorem 12. Two triangles are congruent if their corresponding angles have equal measures.



Proof: Suppose the theorem false. Then there exist Δ ABC and Δ A'B'C' which are not congruent such that $m\angle A = m\angle A'$, $m\angle B = m\angle B'$, $m\angle C = m\angle C'$. Since the triangles are not congruent $AB \neq A'B'$ (otherwise they would be congruent by A.S.A.). Similarly $AC \neq A'C'$ and $BC \neq B'C'$. Consider the triples AB, AC, BC and A'B', A'C', B'C'. One of these triples must contain two numbers which are greater than the corresponding numbers of the other triple. Consequently it is not restrictive to suppose $AB > A'B'$ and $AC > A'C'$.

Then we can find B'' on $\overline{AB}$ such that $A'B' = AB''$ and C'' on $\overline{AC}$ such that $A'C' = AC''$. It follows that $\Delta AB''C'' \cong \Delta A'B'C'$ so that

$$m\angle AB''C'' = m\angle B' = m\angle B.$$

Hence $\angle BB''C''$ is supplementary to $\angle B$. Similarly $\angle CC''B''$ is supplementary to $\angle C$. Therefore quadrilateral BB''C''C has an angle measure sum of 360. This contradicts Corollary 9 and our proof is complete.

We have here a striking contrast with Euclidean geometry. In view of Theorem 12, in Lobachevskian geometry there cannot be a theory of similar figures based on the usual definition. For if two triangles were similar, the measures of their corresponding angles would be equal and they would have to be congruent. In general two similar figures would be congruent and so have the same size. In a Lobachevskian world, pictures and statues would have to be life-size to avoid distortion.

Now let us consider the question of measurement of area. For the sake of simplicity we restrict ourselves to triangles. Clearly the Euclidean procedure of measuring area in terms of square units will not apply since squares don't exist in Lobachevskian geometry. To clarify the problem we ask what are the essential characteristics of area. As a minimum we require:

(1) The area of a triangle shall be a uniquely determined positive real number;

(2) Congruent triangles shall have equal areas;

(3) If a triangle T is split into two triangles T_1 and T_2 then the area of T shall be the sum of the areas of T_1 and T_2.

It is easy to verify that the familiar formula for the area of a triangle in Euclidean geometry satisfies these conditions.

There is a similar area formula (or area "function") in Lobachevskian geometry but it is most naturally expressed in terms of the angles of a triangle. To state it formally we introduce the

Definition. The *defect* (or *deficiency*) of ΔABC is $180 - (m\angle A + m\angle B + m\angle C)$.

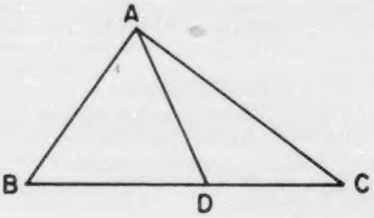
Note that the defect of a triangle literally is the amount by which its angle measure sum falls short of 180.

The defect of a triangle has the essential properties of area:

Theorem 13. Th[illegible]fect of a triangle satisfies properties (1), (2), (3), above.

Proof: Clearly (1) is satisfied since the defect of a triangle is a definite positive number. Property (2) holds since congruent triangles have equal angle sums and so equal defects.

To establish (3) let Δ ABC be given and let D be a point of $\overline{BC}$, so that Δ ABC is split into Δ ABD and Δ ADC. The sum of the defects of the latter two triangles is



$$\begin{aligned} &180 - (m\angle BAD + m\angle B + m\angle BDA) + 180 - (m\angle CAD + m\angle C + m\angle CDA) \\ &= 180 - (m\angle BAD + m\angle CAD + m\angle B + m\angle C) \\ &= 180 - (m\angle BAC + m\angle B + m\angle C) \end{aligned}$$

which is the defect of Δ ABC.

Are there other area functions besides the defect? It is easy to verify that if we multiply the defect by any positive constant k, we obtain an area function which satisfies Properties (1), (2), (3). This is not as remarkable as it might seem, since the specific form of our definition of defect depends on our basic agreement to measure angles in terms of degrees. If we adopt a different unit for the measure of angles and define "defect" in the natural manner, we obtain a constant multiple of the defect as we defined it. To be specific, suppose we change the unit of angle measurement from degrees to minutes. This would entail two simple changes in the above theory: (a) each angle measure would have to be multiplied by 60; (b) the key number 180 would have to be replaced by 60 times 180. Thus the appropriate definition of "defect" would be 60 times the defect as we defined it.

Finally we note that it can be proved that any area function satisfying (1), (2), (3) must be k times the defect (our definition) for some positive constant k. In view of this it is natural to define the area of a triangle to be its defect.

Query. Which of the Properties (1), (2), (3) holds for the defect of a triangle in Euclidean geometry?

It is interesting to note that in Euclidean spherical geometry the sum of the angle measures of a triangle is greater than 180 and the area of a triangle is given by its "excess", that is its angle measure sum minus 180.

Exercise 1. Given Δ ABC with points, D, E, F in $\overline{AB}$, $\overline{BC}$, $\overline{AC}$ respectively. Prove that the defect of Δ ABC is the sum of the defects of the triangles ADF, BED, CFE, and DEF.

Exercise 2. If points P, Q, R are inside Δ ABC prove that Δ ABC has a larger defect than Δ PQR.

We conclude this part by observing that the familiar Euclidean property - parallel lines are everywhere equidistant - fails in Lobachevskian geometry. In fact there are parallel lines of two types. If two parallel lines have a common perpendicular they diverge continuously on both sides of this perpendicular. If two parallel lines don't have a common perpendicular they are asymptotic - that is if a point on one recedes endlessly in the proper direction, its distance to the other will approach zero.

Conclusion

In its further development Lobachevskian geometry is at least as complex as Euclidean geometry. There is a Lobachevskian solid geometry, a trigonometry and an analytic geometry - problems in mensuration of curves, surfaces and solids require the use of the calculus.

You may object that the structure is grounded on sand - that Lobachevskian geometry is inconsistent and eventually will yield contradictory theorems. This of course was the implicit belief that led mathematicians for 2,000 years to try to prove Euclid's Parallel Postulate. Actually we have no absolute test for the consistency of any of the familiar branches of mathematics. But it can be proved that the Euclidean and Lobachevskian geometries stand or fall together on the question of consistency. That is, if either is inconsistent, so is the other.

Once the ice had been broken by Bolyai and Lobachevsky's successful challenge to Euclid's Parallel Postulate, mathematicians were stimulated to set up other non-Euclidean geometries - that is, geometric theories which contradict one or more of Euclid's Postulates, or approach geometry in an essentially different way. The best known of these was proposed in 1854 by the German mathematician Riemann (1826-1866). Riemann's theory contradicts Euclid's Parallel Postulate by assuming there are no parallel lines. This required the abandonment of other postulates of Euclid since we have proved the existence of parallel lines without assuming any parallel postulate (Corollary 3). In Riemann's theory, in contrast to those of Euclid and Lobachevsky, a line has finite length. Actually there are two types of non-Euclidean geometry associated with Riemann's name, one called single elliptic geometry in which any two lines meet in just one point, and a second, double elliptic geometry, in which any two lines meet in two points. The second type of geometry can be pictured in Euclidean space as the geometry of points and great circles on a sphere.

Riemann also introduced a radically different kind of geometric theory which builds up the properties of space in the large by studying the behavior of distance between points which are close together. This theory, called Riemannian Geometry, is useful in applied mathematics and physics and is the mathematical basis of Einstein's General Theory of Relativity.

Bolyai and Lobachevsky have opened for us a door on a new and apparently limitless domain.

MINIATURE GEOMETRIES

1. Preamble. In a given set of postulates for a special part of mathematics, it is hardly to be expected that the laws of classical logic, the rules of grammar and a definition of all the terms be included. We recognize their need but assume them whenever used. We also assume that the reader is familiar with the usual laws of arithmetic and algebra that may be used. Indeed there may be other needed logical assumptions that are overlooked so that the emphasis may be placed upon the particular topic under immediate discussion, and the postulates will be confined to those that have an immediate geometric use.

2. Characteristics of a postulate system. What postulates should we make? There is no definite answer to this question. The answer depends upon the audience and upon the purpose and the preferences (or prejudices) of the individual. However, there are some desirable characteristics of a postulate system, which we proceed to discuss. We may not be able to attain all of them, and may have to make some compromises.

(1) Simplicity.* The postulates should be simple, that is, easily understood by the audience for which they are intended. But simple is a relative term, and depends upon the experience of the audience.

(2) Paucity. It may be desirable to have only a few undefined entities and relations and to make only a few assumptions about them. It may be necessary to sacrifice these characteristics to gain simplicity of understanding. Most texts on plane geometry for beginning students do sacrifice these characteristics, and some texts over-do it to avoid proving converses, especially if the method of proof by contradiction is needed. This puts a high premium on factual geometry as against logical geometry. It is not my purpose here to condemn or commend this

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*See Nelson Goodman, "The Test of Simplicity", Science, October 31, 1958, Vol. 128.

point of view. It all depends upon the audience and the purpose of the text, but it may be very difficult to determine (except by the Rule of Authority) whether the system satisfies the next characteristic.

(3) Consistency. The postulate system should be consistent. It should not be self-contradictory. This part may be easy to determine. For example, we would not want to include two assumptions such as (A): Two lines in the same plane always have a point in common (Projective Geometry) and (B): There are lines in the same plane that have no point in common (Euclidean Geometry). But more is needed. The postulate system should never lead to a contradiction. This may be difficult to determine or impossible to determine. We seldom know all the consequences of the postulate system, and in that case the proof of absolute consistency may not be possible. We content ourselves with relative consistency. If we can give at least one interpretation of the undefined terms based upon our experiences or experiments for which we grant all the assumptions are true, we are satisfied. We call such an interpretation a model. In the case of a simple system such as that for a miniature geometry, the construction of such models may be possible, and indeed in more than one way. In a complex postulate system, such as that needed for all of Euclidean Geometry, logically developed, this may be extremely difficult. If we have more than one model for the same system so that we can find a correspondence connecting every entity and relation of one model with an entity and relation of each of the other models, that is, put the models into one-to-one correspondence, we say the models are isomorphic. We shall do this for some of our miniature geometries. But for more complex geometric systems, we may not have more than one model. The relative consistency of Euclidean Geometry is proved (but it is much too difficult for us to do it) by using arithmetic as a model, and showing it is possible to put Euclidean Geometry into one-to-one correspondence with arithmetic logically developed. Since we have never found a contradiction in arithmetic, we are content to say Euclidean geometry is as consistent as arithmetic. If we wish to prove that a non-Euclidean

geometry is relatively consistent, we find a model (interpretation) within Euclidean geometry for it and after that is done (it is not an easy task and is beyond our intent), we know non-Euclidean geometry is consistent if Euclidean geometry is. This is not the only way it can be done, for arithmetic (algebraic) methods are also available.

(4) Independence. It may be desirable to have all the postulates independent, especially if we are seeking models. By that we mean that the postulate system is such that no postulate can be derived from the others. The arguments present in (2) above are again applicable. In a given postulate system, it may be possible to prove that some of the assumptions could be derived from others, but it may be so difficult that it is a task to be avoided. However, it is not really difficult to prove: "Two distinct lines cannot have more than one point in common" from the assumption: "There is one and only one line that contains two distinct points". The method of contradiction is used, and this points out the essential importance of this method of proof if we wish to make good use of our assumptions of logic. The independence of all the postulates of a system is most readily found in terms of models. If we can find a model that satisfies all but one of the postulates and denies that one, then that particular postulate is independent of the others. If we can do this for each postulate in turn, then the postulates form an independent system.

(5) Completeness: A postulate system for Euclidean geometry, or any other special geometry we wish to discuss, should also be complete. That is, we must include enough postulates to prove all the theorems we wish to prove. This topic will not be discussed in detail here; it is enough to include a warning not to overlook tacit assumptions as Euclid* and his imitators did.

*See Felix Klein, Elementary Mathematics from an Advanced Standpoint; Meserve, The Foundations of Geometry, p. 230-231; Wilder, Foundations of Mathematics, Chapter 1, 2.

We illustrate various ideas mentioned above by confining our attention to incidence properties alone and make no attempt to discuss postulates of measure or separation, but do recognize that parallelism is essentially an incidence property. First we confine our attention to three types of miniature geometries which contain only a finite number of points and lines:
I. A three point - three line geometry; II. A four point - six line geometry; III. A seven point - seven line geometry.

After that we illustrate the incidence properties of Hyperbolic Geometry by considering two models in which the number of points on a line is infinite and where we change the Parallel Postulate from its usual Euclidean form.

3. A three point geometry.

Undefined: point, line, on.

Concerning these undefined terms, we make the following four postulates:

P1. There exist three and only three distinct points.

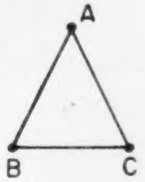
P2. On two distinct points there is one and only one line.

P3. Not all points are on the same line.

P4. On two distinct lines there is at least one point.

As far as consistency is concerned, there does not seem to be any direct contradiction. The relative consistency of the system is accepted on the basis of any one of the following three isomorphic models.

(a) The usual model of a triangle, consisting of three non-collinear points, but here a line contains only two points. The line segments of a more complete geometry are merely drawn to point out the three pairs of points. A line is merely a set of two points. It is easy to observe that Postulates P1 — P4 are all satisfied.



(b) A group of three boys forming committees of two in all possible ways. If the boys are called A, B, C, the committees are the three pairs (A,B), (B,C), (C,A). If the postulates are read with 'boy' replacing 'point', 'committee' replacing 'line' and 'member of' replacing 'on', with possible changes in language to preserve the meaning, it is easy to see P1, P2, P3 are obviously satisfied by the way the committees were formed. A simple observation of the three committees checks P4.

(c) Points are interpreted as the special ordered number triples (x,y,z): $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$. Lines are interpreted as the special equations $x = 0$, $y = 0$, $z = 0$. A 'point' is 'on' a 'line' if its coordinates satisfy the equation of the line.

P1 follows from our choice of coordinates.

P2 must be verified: $A(1,0,0)$ and $B(0,1,0)$ are both on $z = 0$ but not both are on $x = 0$ or $y = 0$. A similar verification is needed for the other pairs of points.

P3: The point $A(1,0,0)$ does not satisfy the equation of the line $\overleftrightarrow{BC}$, $x = 0$.

P4: There are three distinct pairs of lines (i) $x = 0$, $y = 0$; (ii) $y = 0$, $z = 0$; (iii) $z = 0$, $x = 0$. It is easy to verify that $C(0,0,1)$, $A(1,0,0)$, $B(0,1,0)$ lie on the pairs (i), (ii), (iii) respectively.

We prove three theorems directly from the postulates without a model. For heuristic purposes any one of the models could be used.

Theorem 1. On two distinct lines there is not more than one point.

Proof: If two lines had two distinct points in common, then Postulate P2 would be contradicted. Hence Theorem 1 is true.

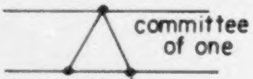
Theorem 2. There exist three and only three lines.

Proof: Since there are three and only three points (P1), there are only three pairs of points: (A,B), (B,C), (C,A). Each such pair determines one and only one line (P2). These lines are all distinct (P3). Hence there are three and only three lines.

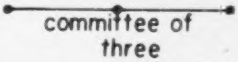
Theorem 3. Not all lines are on the same point.

Proof: There are three and only three lines (A,B), (B,C), (C,A), (Theorem 2). The first and third are on the point A, but this point is not on the line (B,C) because of P3. A similar argument concerning the points B and C completes the proof.

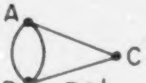
Of course all three of these theorems could have been verified in any model. That is, we could have taken them as postulates too, but then the system would not have been an independent one. To demonstrate the independence of the original system P1 - P4 we use geometric models but either of the other models could be used equally as well. We use the notation P4' to indicate that P4 is denied but P1, P2, P3 are satisfied. Similar meanings are given to P3', P2', and P1'. The model P4' is constructed by adding a fourth line (denying Theorem 2) in such a way that there are two lines which have no point in common. This denies P4, but the other postulates are satisfied. In the model P3', all three points are on the same line and the other postulates may be verified. In the model P2', there are two lines which contain both A and B. In terms of the committee interpretation you may think of A and B both being on two distinct committees, say the Finance Committee and the Custodian Committee. The model for P1' is not shown here. It must contain more than three points.



P4'



P3'



P2'

The smallest such model which will also satisfy the other axioms is the model for a seven point geometry to be discussed in Section 5. After that model is presented the proof of the independence of the system P1 - P4 will be complete.

4. A four point geometry. Again point, line, and on are undefined. To distinguish the postulates from those just used we use the letter Q.

Q1. There exist four and only four distinct points.

Q2. On two distinct points there is one and only one line. (P2)

Q3. Every line contains two and only two points.

Theorem 1. There exist six and only six lines.

Proof: The number of pairs of points is the number of combinations of four things taken two at a time, ${}_4C_2 = \frac{4 \cdot 3}{2} = 6$ (Q1) and this is the number of lines (Q2). These lines are all distinct, (Q3). Hence the theorem is proved.

If we call the points O, A, B, C, the lines are represented by the point pairs (O,A); (O,B); (O,C); (A,B); (A,C); (B,C).

Definition. Two lines are parallel if they have no point in common.

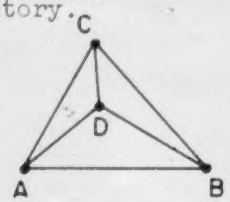
Note that the word parallel is used in a very special sense. No concept of a plane has yet been introduced.

Theorem 2. Through a given point not on a given line there is one and only one line parallel to the given line.

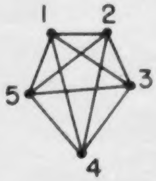
Proof: A given point, say A, lies on three and only three lines and these lines are distinct (Q1, Q2, Q3). If we pick one of these lines, say AO, neither of the remaining points, B and C, can lie on it (Q3), and hence the two lines have no point in common and so are parallel by definition.

Several models of this geometry are available. The two-member committee model is quite apparent. Each member is on three committees but there is always a unique second committee that can meet while this member is engaged in committee business.

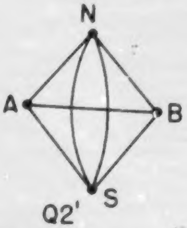
In order to present geometric models, we imagine the model to be embedded in ordinary Euclidean geometry and then abstract from the diagram those features that are wanted. One such model is that of a complete quadrangle (a term borrowed from projective geometry) which consists of four points, no three collinear, and the six lines which they determine by pairs. Of course you must recognize that our line is only a point-pair. It is easy to verify that Postulates Q1, Q2, Q3 are all satisfied. Models Q1', Q2', Q3', needed to prove the postulates are independent, are more or less self-explanatory.



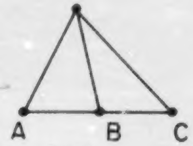
Q1, Q2, Q3



Q1'



Q2'



Q3'

If the model Q2' bothers you, think of it in terms of a diagram drawn on a sphere with N and S being the poles, or if you know something of chemical bonds, think of it in terms of a double bond between N and S, and all the rest as single bonds.

The figure for Q1, Q2, Q3 could be imagined in ordinary 3-space thus forming a tetrahedron. Indeed we could then add additional postulates.

Undefined: plane.

Q4. On three points there is one and only one plane.

If we think entirely in terms of plane geometry each of the models already drawn also satisfy Q4.

Q5. Every plane contains three and only three points.

None of the models of plane geometry satisfy this axiom, which, however, is satisfied by the tetrahedron model. That is, the tetrahedron model satisfies all five postulates Q1 - Q5. It is possible to present models in 3-space to prove the independence of these five postulates but this will not be done here, but the reader is urged to try his hand at it.

Another property of the tetrahedron model that the reader may be interested in proving is that it satisfies Incidence Postulates 1, 6, 7, 8, and Existence Postulate 5 of our text.

The committee interpretation of this enlarged system takes into account three-member committees as well as two-member committees. Our tetrahedron model is for a four point - six line - four plane geometry.

Let us return to the system Q1, Q2, Q3 and its two geometric interpretations and discuss algebraic systems isomorphic to them. For the complete quadrangle model, we consider points as the special ordered number triples (x,y,z): $A(1,0,0)$; $B(0,1,0)$; $C(0,0,1)$; $O(1,1,1)$. As lines we take the six equations $x = 0$, $y = 0$, $z = 0$, $x = y$, $y = z$, $z = x$. We say a point is on a line if its coordinates satisfy the equation of the line.

Q1 is satisfied by the way coordinates were introduced. It is now possible to verify Q2 and Q3. There are six pairs of points and it is possible to show that any pair lies on one and only one line and this line contains neither of the other points. For example, $B(0,1,0)$ and $C(0,0,1)$ satisfy the equation $x = 0$, but neither $A(1,0,0)$ nor $O(1,1,1)$ do; $B(0,1,0)$ and $O(1,1,1)$ satisfy the equation $x = z$, but neither of the points $A(1,0,0)$ or $C(0,0,1)$ do. Similarly, for the four other pairs.

For the tetrahedron model, we consider points as the special ordered number triples (x,y,z): $A(1,0,0)$; $B(0,1,0)$; $C(0,0,1)$ and $O(0,0,0)$. (Note the difference between the two models.) As the lines we consider the six pairs of equations which can be formed from the four equations $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$.

(These are the equations of the four planes.) Q1 is satisfied by the way coordinates were introduced. It is now possible to verify Q2 and Q3. For example, B(0,1,0) and C(0,0,1) satisfy the two equations $x = 0$, $x + y + z = 1$, but *both* do not lie on either $y = 0$ or $z = 0$. A similar analysis can be given for every other pair of points. In this algebraic model, Postulates Q4 and Q5 also may be verified.

5. *A seven point geometry*. As mentioned earlier this geometry is one that denies the existence of only three points but satisfies P2, P3, P4 of the three-point geometry. We repeat these postulates for convenience of reference. The essential distinction between this geometry and those already discussed is that every line contains three and only three points. It is necessary to include a postulate which guarantees there is at least one line.

Undefined: *point*, *line*, *on*.

P2. On two distinct points there is one and only one line.

P3. Not all points are on the same line.

P4. On two distinct lines there is at least one point.

P5. There exists at least one line.

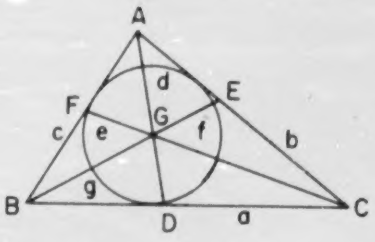
P6. Every line is on at least three points.

P7. No line is on more than three points.

Of course P6 and P7 could be put together to say: Every line is on three and only three points.

We construct a special model for this postulate system by selecting seven distinct points, which we call A, B, C, D, E, F, G. We define seven and only seven lines, a, b, c, d, e, f, g, each being a set of three points, by means of the following table.

A	B	C	D	E	F	G
B	C	E	A	G	D	F
F	D	A	G	B	E	C
c	a	b	d	e	g	f



It is not our purpose to discuss the many theorems that can be proved from this postulate system, but to point out several interpretations of it. It may bother you a bit to call (D,E,F) a line, but it is a line by definition just as much as the triple (A,B,F) is a line. Of course this geometry is not like the Euclidean geometry of your experience -- it is a finite projective geometry where we have considered only incidence properties. However, its interpretation as a group of seven persons and seven committees of three and only three members is also available. Since we set up the model by definition (committee aspect) and then drew a diagram to correspond, we must verify all the Postulates P2 to P7. This may be long in detail but it is not difficult. There are 21 pairs of points $({}_7C_2 = \frac{7\cdot 6}{2})$ and 21 pairs of lines, but an examination of the table shows that each row contains each letter once and only once, and each letter is in three and only three columns, and this will simplify the details. It is merely time consuming to verify all the postulates; these postulates are all satisfied in the geometric model. To verify P4, for example, from the table, it is necessary to consider 21 pairs of lines, and indeed it is easy to verify not only that each pair has a point in common (there are no pairs of parallel lines) but only one point in common.

The results can be tabulated as follows

c	a	b	d	e	g	f
b	e	f	a	g	c	d
d	c	a	g	b	f	e
A	B	C	D	E	F	G

Not only may we verify P2 - P7 in this way, but also the dual of each of these statements. The dual is obtained by interchanging the words point and line wherever they appear. For example, the dual statement to P6 and P7 combined would read:

D6, 7. Every point is on three and only three lines.

This is easily verified from the defining table.

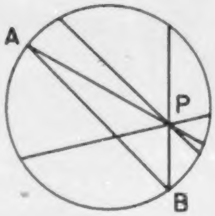
The algebraic isomorphism for this geometry consists of the following assignments of coordinates to points and equations to lines:

A(1,0,0); B(0,1,0); C(0,0,1); D(0,1,1); E(1,0,1); F(1,1,0);
G(1,1,1); a: $x = 0$; b: $y = 0$; c: $z = 0$; d: $y = z$;
e: $x = z$; f: $x = y$; g: $x + y + z = 2$.

All the postulates could be verified purely algebraically. For example, D(0,1,1) and E(1,0,1) both lie on the line $x + y + z = 2$, but not both are on any other line. The line $x = y$ contains the three points C(0,0,1), G(1,1,1), F(1,1,0) but no other point. This is enough to give the general idea.

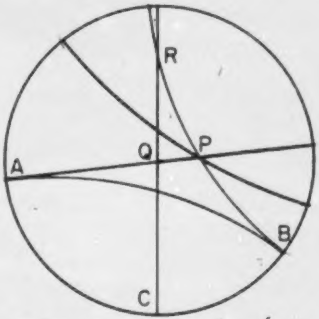
6. Models for a hyperbolic geometry. In order to discuss such a model, it will be embedded in a Euclidean plane. Hence we assume that the postulates of Euclidean geometry as stated in the text have been made and Euclidean geometry has been developed. We will use the terms point, line, plane, and circle as developed in such a treatment. The corresponding words placed in quotes will stand for entities in a new geometry, and will be defined by means of Euclidean terms. In this way we will obtain models to illustrate some of the incidence properties of hyperbolic geometry.

The first model is often called a projective model, but the explanation of the term is beyond our present means. Consider a circle. We define a "point" of our new geometry to be a point in the interior of the circle; a "line" is a chord of this circle without its end-points; the "plane" is the interior of the circle. It is easy to observe that two "lines" may or may not intersect. If two chords of the circle intersect on the circle, we say that the corresponding "lines" are "parallel". Note that there is a definite distinction between two "lines" being "parallel" and two "lines" not intersecting. It is



also easy to observe that through a given "point" P, there are exactly two "lines", $\overline{PA}$ and $\overline{PB}$, which are "parallel" to the "line" $\overline{AB}$, and that there are an infinite number of "lines" through P that do not intersect the "line" $\overline{AB}$.

In the above model length and angular measure are distorted, and a study of projective geometry is needed to discuss the model. There is a model, called *Poincaire's Universe*, where length is distorted but angular measure is not (but no proof is intended). To understand this model some knowledge of orthogonal circles in Euclidean geometry is required, and the corresponding theorems are not usually presented in an introductory course in plane geometry. We state the necessary definitions and theorems (without proof).



Two circles are *orthogonal* if their angle of intersection is a right angle. By the angle of intersection of two circles we mean the angle between the tangent lines drawn at a common point.

Through two points there is one and only one circle (or line) orthogonal to a given circle.

In the Poincaire model, a "point" is again a point inside a given circle C, and the "plane" is the set of all points in the interior of the circle. A "line" is either a diameter of the circle C, without its end-points, or that part of a circle orthogonal to the circle C which lies inside C. We note, therefore, that through two "points" there is one and only one "line". Two "lines" are said to be "parallel" if their corresponding diameters or circles intersect on C. It is again easy to observe that through a given "point" P, there are two "lines" $\overset{\frown}{PA}$ and $\overset{\frown}{PB}$, which are "parallel" to the "line" $\overset{\frown}{AB}$, and that there are an infinite number of "lines" through P that do not intersect the "line" $\overset{\frown}{AB}$. One more idea may be observed in this diagram (based on the assumption that angular measure is not distorted).

"The sum of the measures of the "angles" of a "triangle" such as Δ PQR or Δ APB is less than 180."

A more detailed study of the geometry of the circle in the Euclidean plane, including a study of the concept of cross-ratio is needed to carry the discussion further. Some further results and suggestions or indications of ideas that might be investigated can be found in Eves and Newson, Introduction to Foundations and Fundamental Concepts of Mathematics.

AREA

It is possible to develop the theory of area, as far as we need it, from a very simple set of postulates, which are intuitively acceptable. In some respect they are more intuitive than the ones given in the text, being simpler to state and requiring fewer preliminary definitions. For example, it is not necessary to define polygonal region in order to state the postulates. It is satisfying that this is one of the many cases in mathematics in which intuition and rigor go hand in hand. We shall sketch this development at least up to the point where it is clear that we could proceed as in the text, by deriving as theorems the postulates of the text which are not already included in our set. Some of the early theorems may appear obvious and hardly worth proving; but if we recognize the fact that postulate systems are constructed by fallible humans and need to be tested by their consequences, then we should derive satisfaction from the provability of some "obvious" statements by means of our postulate system.

We always speak of the area of something, and this something is a region or a figure -- which are simply names for certain sets of points in a plane. Thus, area is a function of sets, an assignment of a unique real number to a set. Whenever we speak of a function, it is important to be quite clear as to the domain of the function, that is, the set of objects for which the function provides us with an answer. In our case, we must ask, what sets are to have an area assigned to them? We could limit ourselves, if we wished, to simple sets, like polygonal regions. This has the disadvantage that it eliminates regions bounded by circles, ellipses, hyperbolas, and other smooth curves, regions which (our intuition tells us) should have areas. Of course, we do not want huge sets like the whole plane, or half-planes, or the interiors of angles, to have area. These all have the property of being unbounded. Fortunately, it can be proved that it is possible to assign a reasonable area to every reasonable set in the plane. The first "reasonable" means that the area function will not violate our intuition. The second "reasonable" we shall interpret in the widest possible sense, namely, as "bounded". A bounded set is one that can be enclosed in some square (or circle). We shall therefore adopt as our first area postulate the following:

Postulate A1. There is a function A (called area) defined for all bounded sets in the plane; to each bounded set S, A assigns a unique non-negative number A(S).

Let us observe immediately that a point and a segment are bounded sets, so we have committed ourselves to the unfamiliar position of attributing an area to such sets. The area will turn out to be zero, of course. There are excellent precedents: let us recall that we have allowed ourselves to speak of the distance from a point to itself as being zero. Analogously, in the theory of probability it is useful to have events with zero probability, even though the events are possible. Indeed, the theories of linear measure, area, volume, probability, and counting all have a great deal in common, since they are concerned with assigning measures to various sets. Far from being a disadvantage, the concept of zero area is extremely valuable. It makes explicit our sound intuition of what sets are "negligible" as far as area is concerned. For example, the Area Addition Postulate in the text (Postulate 19) essentially asserts that the area of the union of two sets is equal to the sum of their areas, provided that they overlap in a "negligible" set -- a finite union of points and segments. It is somewhat easier to accept an Area Addition Postulate in which the "negligible" set is the empty set, as in Postulate A2 that follows, and to prove later that certain sets really are "negligible".

Postulate A2. If S and T are bounded sets in the plane which have no points in common, then the area of the union of S and T is equal to the sum of the areas. That is, if V is the union of S and T, then $A(V) = A(S) + A(T)$.

We have already remarked that Postulate A2 is weaker in one respect than the Area Addition Postulate in the text, for it does not allow even one point in common to the sets S and T. Observe also that Postulate A2 does not need to assert the existence of A(V). This is in fact a simple consequence of Postulate A1, for the union of two bounded sets is also bounded.

Our third postulate will give the essential connection between our geometry and area. For this we need a somewhat more general concept of congruence than the usual one. Two sets will be called congruent if there is a one-to-one correspondence between them which preserves all distances. More precisely, suppose there is a one-to-one correspondence between S and T such that, A and B being any points of S corresponding to A' and B' in T, the distance AB is equal to the distance A'B'. Then we shall say that S is congruent to T, or $S \cong T$. Our definitions of congruence for segments, angles, triangles, and circles are special cases of this more general definition. For a fuller treatment, see the Appendix on Rigid Motion and the Talk on Congruence. If our area function is to be reasonable, then congruent sets should have the same area:

Postulate A3. If S is a bounded set and $S \cong T$, then $A(S) = A(T)$.

Again, it is easy to see intuitively that if S is bounded and $S \cong T$, then T is bounded, and A(T) exists by Postulate A1.

Now let us consider the area of a square of side 1 together with its interior. For all we know from the first three postulates, this area might be 0. This does violence to our intuition, and even more, we could then prove that every bounded set has area 0. Therefore we must postulate that this area is positive, say equal to k. But then the new area function defined by $A'(S) = \frac{1}{k}A(S)$ would be just as good as the old and would have the desirable property that it assigns the value 1 to the unit square and its interior. We shall therefore postulate this immediately:

Postulate A4. If S is the set consisting of a square of side 1 together with its interior, then $A(S) = 1$.

This postulate essentially does no more than (a) rule out the trivial case of a constantly zero area function, and (b) fix the unit by which we measure the area of a set. We can think of it as a normalization postulate, and shall speak of our area function as being normalized.

Summing up our four postulates -- these are all we need -- we see that we have a non-negative (Postulate A1), finitely-additive (Postulate A2), normalized (Postulate A4) function of bounded sets in the plane (Postulate A1), invariant under rigid motion (or congruence) (Postulate A3). The term "finitely-additive" refers to the fact that we can easily replace the two sets in Postulate A2 by any finite number of sets, no two of which have a point in common.

At the beginning of this talk, we stated that it is possible to assign a reasonable area to every reasonable set in the plane. This theorem, asserting the existence of such a function, is rather deep and difficult to prove. Nevertheless, it provides us with a sound basis for a treatment of area in the plane. The set of four postulates matches our intuition quite well, especially if we have not subjected to close scrutiny the vast generality involved in the phrase "all bounded sets in the plane". It should be remarked that the theorem does not guarantee a unique function, but any two functions that satisfy the conditions will agree for decent, non-pathological sets such as polygonal regions, circular regions, and regions bounded by arcs of smooth curves like parabolas, hyperbolas, ellipses, etc.

It would be pleasant if this treatment could be generalized to volume in three dimensions. Surprisingly, the corresponding statement in three dimensions is false. One form of the Banach-Tarski Paradox asserts that it is possible to split each of two spheres of different radii into the same finite number of sets, corresponding sets from each sphere being congruent. If the three-dimensional statement were true, the corresponding sets would have equal volumes, by the invariance under congruence, and therefore the spheres would have equal volume, by the finite-additivity of volume. On the other hand, the usual formula for the volume of a sphere would be valid, thus leading to a contradiction. In three-dimensions, therefore, it is necessary to limit our volume function to a more restricted class of sets than the bounded ones. This restriction is no cause for alarm, since the resulting domain of the volume function is still much wider than we need for ordinary purposes. The sets that we exclude are all really "wild". With this one modification the methods used here are still applicable in three-dimensions.

Now we shall proceed with the business of developing the consequences of our set of postulates. These consequences we shall state as theorems. First, however, we need a simple result which has nothing directly to do with area, but which is a basic property of our real number system.

Theorem 1. If a is a non-negative number such that for every positive integer number n, $na \leq 1$, then $a = 0$.

The statement may seem a little strange, but it is specifically designed to yield the type of result needed, namely that a certain number is 0. For example, suppose that we wish to prove that a certain formula yields the correct value for the area of a given figure. Let the area be A and the number given by the formula be B. Denote by a the absolute value of their difference, $|A-B|$. Then we wish to prove that $a = 0$. We may be able to show that no multiple of a exceeds 1. If so, then Theorem 1 assures us that $a = 0$ and therefore that $A = B$. Another way of stating Theorem 1 is: There is no positive number which is simultaneously $\leq 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$. Still another way is: Every positive real number is less than some positive integer. If we regard this last statement as being a known property of real numbers, then the proof of Theorem 1 is quite easy. Suppose, indeed, that a satisfies the hypotheses of the theorem, but that $a > 0$. Then $\frac{1}{a}$ is a positive number, and there is a positive integer n such that $\frac{1}{a} < n$, by what we have just said. For this n, $1 < na$, contradicting the hypothesis $na \leq 1$. Therefore the assumption $a > 0$ is false. Since $a > 0$ or $a = 0$ by hypothesis, and the first is false, the second must be true.

We can now prove some rather obvious results which are usually assumed implicitly in customary treatments. They are, in fact, somewhat less obvious than some of the theorems that Euclid took the trouble to prove (e.g., the theorem that vertical angles are congruent). It is interesting to contemplate what the situation might have been if Euclid had decided that these were worthy of statement and proof. Perhaps school boys for centuries would have studied and proved:

Theorem 2. The area of a point is 0.

Proof: Let S be a unit square plus its interior. By Postulate A4, $A(S) = 1$. Let n be an arbitrary positive integer, and choose n points $P_1, P_2, \ldots, P_n$ in S. If T is the set $\{P_1, \ldots, P_n\}$, then by Postulate A2 (rather, by the generalization of Postulate A2 to n disjoint sets), we have $A(T) = A(P_1) + A(P_2) + \ldots + A(P_n)$. Now any two one-point sets are congruent, so by Postulate A3, $A(P_1) = A(P_2) = \ldots = A(P_n)$, and $A(T) = nA(P_1)$. Let R be all of S except for the points of T. Then R and T have no points in common and their union is S. By Postulate A2,

$$A(T) + A(R) = A(S).$$

By Postulate A1, $A(R) \geq 0$. Therefore

$$A(T) \leq A(S).$$

Substituting 1 for $A(S)$ and $nA(P_1)$ for $A(T)$, we get

$$nA(P_1) \leq 1.$$

In Theorem 1, we may take $a = A(P_1)$, since $A(P_1)$ is non-negative by Postulate A1. Therefore $a = 0$, that is, $A(P_1) = 0$. Since every point is congruent to P, $A(P) = 0$ for every point P, by Postulate A3.

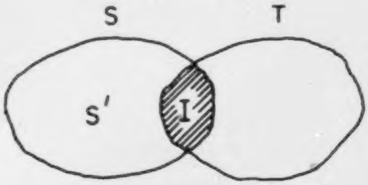
Observe that in the proof of Theorem 2, we proved and made use of a special case of:

Theorem 3. If T is a subset of the bounded set S, then $A(T) \leq A(S)$.

The proof may be left to the reader.

Now we state a useful theorem which is similar to Postulate A2, but which has a weaker hypothesis.

Theorem 4. If S and T are bounded sets, V is the union of S and T, and I is the intersection of S and T, then $A(V) = A(S) + A(T) - A(I)$.



Proof: Let S' be the part of S not in T. Then the union of S' and I is S, and S' and I are disjoint. By Postulate A2,

$$A(S) = A(S') + A(I).$$

Also, the union of S' and T is V, and S' and T are disjoint. By Postulate A2,

$$A(V) = A(S') + A(T).$$

Therefore

$$\begin{aligned} A(V) &= A(S) = A(I) + A(T) \\ &= A(S) + A(T) - A(I). \end{aligned}$$

Theorem 5. If S and T are bounded sets and V is their union, then

$$A(V) \leq A(S) + A(T).$$

The proof follows from Theorem 4 on observing that $A(I) \geq 0$, by Postulate A1.

Theorem 6. If S_1, S_2, ..., S_n are bounded sets and V is their union, then

$$A(V) \leq A(S_1) + A(S_2) + \dots + A(S_n).$$

The proof follows from Theorem 5 by induction.

Next, we prove another "obvious" theorem.

Theorem 7. The area of a segment is 0.

Proof: Let $\overline{BC}$ be a given segment, of length k. There is a natural number m such that $k \leq m$. On the ray $\overrightarrow{BC}$, let D be the point such that $BD = m$. To prove that $A(\overline{BC}) = 0$ it is sufficient to show that $A(\overline{BD}) = 0$, by Postulate A1 and Theorem 3. Now $\overline{BD}$ is the union of m segments S_1, ..., S_m of length 1. These segments are not disjoint, but we can still apply Theorem 6 to get

$$\begin{aligned} A(\overline{BD}) &\leq A(S_1) + \dots + A(S_m) \\ &= mA(S_1), \end{aligned}$$

since S_1, ..., S_m are all congruent. Therefore it is sufficient to show that a segment of length 1 has area 0. The proof of this proceeds as in Theorem 2, by fitting an arbitrary number n of disjoint unit segments within a unit square. We omit the details.

We are now in a position to prove that the boundary of a polygonal region (defined in Chapter 11) has no influence on its area.

Theorem 8. Let R be a polygonal region and let R' be the same region with all or part of the boundary removed. Then $A(R') = A(R)$.

Proof: Let R_0 be the region R with all of the boundary removed. Then R_0 is contained in R' and R' is contained in R. Therefore

$$A(R_0) \leq A(R') \leq A(R),$$

by Theorem 3. It is sufficient to show that $A(R_0) = A(R)$. Let B be the boundary, consisting of a finite number of segments. By an application of Theorem 6, Theorem 7, and Postulate A1, we find that $A(B) = 0$. But R is the union of the disjoint sets R_0 and B, so

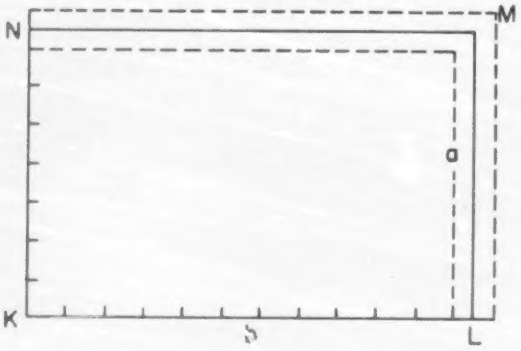
$$A(R) = A(R_0) + A(B) = A(R_0),$$

and the proof is complete.

Postulate 19 of the text now follows readily, since the overlap of the two regions R_1 and R_2 consists of a finite number of points and segments, and the area of the overlap is 0. We state Postulate 19 as a theorem, but omit the proof.

Theorem 9. Suppose that the polygonal region R is the union of two polygonal regions R_1 and R_2, which intersect at most in a finite number of segments and points. Then $A(R) = A(R_1) + A(R_2)$.

Now consider a rectangle R of base b and altitude a. We are aiming at a proof that $A(R) = ab$, this being Postulate 20 of the text.



Choose an arbitrary positive integer n, and determine p and q, also positive integers, by the conditions

$$\frac{p-1}{n} < b \leq \frac{p}{n},$$

$$\frac{q-1}{n} < a \leq \frac{q}{n}.$$

Starting at K, lay off p segments of length $\frac{1}{n}$ along ray $\overrightarrow{KL}$ and q segments of length $\frac{1}{n}$ along ray $\overrightarrow{KN}$. Then L is on the p-th segment on $\overrightarrow{KL}$ and N is on the q-th segment on $\overrightarrow{KN}$. The rectangular region R is now enclosed between two rectangular regions S and T, where S has dimensions $\frac{p-1}{n}$ and $\frac{q-1}{n}$, T has dimensions $\frac{p}{n}$ and $\frac{q}{n}$. Therefore

$$A(S) \leq A(R) \leq A(T).$$

Now S consists of $(p-1)(q-1)$ square regions of side $\frac{1}{n}$, and T consists of pq square regions of side $\frac{1}{n}$. If the area of one of these square regions is A_n, then

$$A(S) = (p-1)(q-1)A_n,$$

$$A(T) = pqA_n,$$

so

$$(p-1)(q-1)A_n \leq A(R) \leq pqA_n.$$

It remains to compute A_n and then $A(R)$. But a unit square, whose area is 1, can be split up into n^2 squares of side $\frac{1}{n}$,

$$1 = n^2A_n,$$

$$A_n = \frac{1}{n^2}$$

Therefore

$$(p-1)(q-1)\cdot\frac{1}{n^2} \leq A(R) \leq pq\cdot\frac{1}{n^2}.$$

Now, from the conditions determining p and q.

$$\frac{p-1}{n}\cdot\frac{q-1}{n} \leq ab \leq \frac{p}{n}\cdot\frac{q}{n}.$$

The two fixed numbers $A(R)$ and ab both lie in the interval with end-points $\frac{p-1}{n} \cdot \frac{q-1}{n}$, $\frac{p}{n} \cdot \frac{q}{n}$, so the absolute value of their difference is at most equal to the length of the interval:

$$|A(R) - ab| \leq \frac{p}{n} \cdot \frac{q}{n} - \frac{(p-1)}{n} \cdot \frac{(q-1)}{n},$$

$$|A(R) - ab| \leq \frac{p - q - 1}{n^2}.$$

Since $\frac{p}{n^2}$ is approximately $\frac{b}{n}$ and $\frac{q}{n^2}$ is approximately $\frac{a}{n}$, the right side is approximately $\frac{1}{n}(a + b)$, which is very small if n is large. An application of Theorem 1 to the fixed non-negative number $\frac{|A(R) - ab|}{(a + b)}$ would then yield that this number is 0. To make this argument precise, choose n so large that $\frac{1}{n} \leq a$ and $\frac{1}{n} \leq b$. Then $\frac{p-1}{n} < b$ implies that

$$\frac{p}{n} < b + \frac{1}{n} \leq 2b,$$

and $\frac{q-1}{n} < a$ implies that

$$\frac{q}{n} < a + \frac{1}{n} \leq 2a.$$

Therefore

$$\frac{p + q - 1}{n^2} < \frac{p + q}{n^2} = \frac{1}{n}\left(\frac{p}{n} - \frac{q}{n}\right) \leq \frac{2a + 2b}{n}.$$

Combining this with our previous inequality, we get

$$|A(R) - ab| \leq \frac{2a + 2b}{n},$$

or

$$n \cdot \frac{|A(R) - ab|}{2a + 2b} \leq 1,$$

for all sufficiently large positive integers n, and therefore for all n. By Theorem 1,

$$\frac{|A(R) - ab|}{2a + 2b}$$

is 0, so $A(R) = ab$. This completes the proof of:

Theorem 10. The area of a rectangle is the product of its base and altitude.

We have now reached our goal of establishing Postulates 17-20 of the text from our system of Postulates A1-A4. This may not seem like a great accomplishment if we are interested in polygonal regions only, but it permits the evaluation of areas of other regions without the necessity of making ad hoc extensions of the domain of the area function at a later stage. It provides us with an excellent example of the power of deductive reasoning. Finally, the transition from here to the integral calculus is a smooth and natural one. For example, the calculation of the area under the curve $y = x^n$, for all integers n (including $n = -1$) can be carried out on the basis of this development, without any reference to the differential calculus.